

Supplementary Material

Lean4PHYS: Comprehensive Reasoning Framework for College-level Physics in Lean4

ICLR 2026 Submission #7797

Overview

This supplementary material is submitted in response to the conditional acceptance requirements specified in the Meta Review. To provide evidence that the benchmark *LeanPhysBench* as a part of **Lean4PHYS** framework does not violate copyright and provide documentation suggesting substantial differences between the formalized physics problems in the dataset and source material.

The college-level statements are formalized from the physics problems in the university textbook.

Hugh D. Young and Roger A. Freedman. University Physics with Modern Physics, Global Edition. Pearson Education, 15th edition, 2019. ISBN 9781292314815

The competition-level statements formalized from the physics Olympiad practice book.

Yousheng Shu, Wangyu Hu, and Bingqian Chen. Selected Advanced Physics Problems, Volume II. Higher Education Press, Beijing, 1999. ISBN-13: 9787040069990, ISBN-10: 7040069997.

Here we provide ten examples for the above two difficulty levels as well as two different sources respectively. For each example in College-Level, we provide the original textbook content and the corresponding Lean code. For each example in Competition-Level, we provide the original Chinese version, a direct English translation, our adapted English version for formalization, and the corresponding Lean code. Among these ten cases, three appeared (either in English or in Lean) in our paper, and we have indicated the position of each such problem.

As the examples shown, the formalized physics problems are fundamentally different from the Chinese or English expressions in the source materials, where underlying physics logic and implicit conditions of the nature language question answering problems are extracted to form formal hypotheses and proof goals.

Competition-Level Examples

第一章 运动学

【题1】 如图1-1-1, 直角三角形 ABC 在铅垂平面内, 斜边 AC 与水平边 BC 的夹角为 α , 一质点从 A 点由静止出发, 在重力作用下到达 C 点. 当所循路径为从 A 经 AB 和 BC 时, 从 A 到 B 需时 t_1 , 从 B 到 C 需时 t_2 ; 当所循路径为 AC 时需时 t_3 . 假定质点转折时不花费时间, 且只改变速度方向而速度大小不变.

1. 为使循上述两条路径由 A 到 C 所需时间相等, 即使 $t_1 + t_2 = t_3$, 试问角 α 应为多大?
2. 设角 α 取上述值, 设质点只能在三角形范围内沿竖直路径和水平路径从 A 点到达 C 点, 显然有无穷多种选择. 试问什么路径花费的时间最多? 什么路径花费的时间最少? 所需的最长时间与最短时间之比是多少?

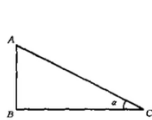


图1-1-1

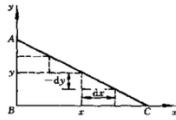


图1-1-2

【分析】 质点在 AB 段和 AC 段均作匀加速直线运动, 前者的加速度为 g , 后者的加速度为 $g \sin \alpha$, 在 BC 段作匀速直线运动. 运用运动学公式可找到各段所需时间之间的关系, 再根据 $t_1 + t_2 = t_3$ 的要求即可确定相应的特定角度 α .

如图1-1-2, 符合题意的路径除了 AC 路径外, 还有由许多水平段以及把各水平段连接起来的各竖直段所构成的各种路径. 符合题意的各种路径中, 各竖直段长度之和等于直角边 AB , 各水平段长度之和等于直角边 BC . 由于质点转折时不花费时间且速度大小也不改变, 故走完各竖直段所需的总时间 t_g 就等于直接从 A 经 AB 到达 B 所需的时间 t_1 . 但走完各水平段所需的总时间则与质点在各水平段的速度大小有关, 即与各水平段的高度有关. 如果选择的路径, 使各水平段均位于三角形的底部 (即为 BC 边), 则质点在该水平段具有的速度最大, 所需时间应最短. 反之, 如果选择的路径, 使各水平段位于尽可能高的位置 (当然, 以不超出边线 AC 为条件), 则质点在各该水平段具有的速度最小, 所需时间应最长. 所以, 需时最长的路径为图1-1-2中虚线所示的锯齿状路径, 各锯齿的水平段 dx 和竖直段 dy 均为无穷小量, 各锯齿紧靠着 AC 而又都不超出 AC 边. 确定了需时最短和最长的路径, 求出相应的最短和最长时间, 即可得出两者之比.

【解】 1. 设 $AC = L$, 由运动学公式, 有

第一章 运动学

[Problem 1] As shown in Figure 1-1-1, right triangle ABC lies in a vertical plane, with the angle between the hypotenuse AC and the horizontal side BC . A particle starts from rest at point A and reaches point C under the influence of gravity. When the path taken is from A through AB and BC , it takes time t_1 to travel from A to B and time t_2 to travel from B to C ; when the path taken is AC , it takes time t_3 . Assume that the particle does not spend time turning and only changes the direction of its velocity while keeping its magnitude unchanged.

1. To make the time required to travel from A to C equal along the two paths mentioned above, even if $t_1 + t_2 = t_3$, what should the angle be?

2. Assuming the angles take the values mentioned above, and that a particle can only reach point C from point A along vertical and horizontal paths within the triangle, there are clearly infinitely many choices. Which path takes the longest? Which path takes the shortest? What is the ratio of the longest time to the shortest time?

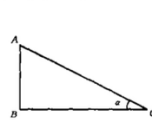


图1-1-1

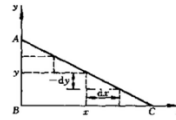


图1-1-2

【分析】 质点在 AB 段和 AC 段均作匀加速直线运动, 前者的加速度为 g , 后者的加速度为 $g \sin \alpha$, 在 BC 段作匀速直线运动. 运用运动学公式可找到各段所需时间之间的关系, 再根据 $t_1 + t_2 = t_3$ 的要求即可确定相应的特定角度 α .

如图1-1-2, 符合题意的路径除了 AC 路径外, 还有由许多水平段以及把各水平段连接起来的各竖直段所构成的各种路径. 符合题意的各种路径中, 各竖直段长度之和等于直角边 AB , 各水平段长度之和等于直角边 BC . 由于质点转折时不花费时间且速度大小也不改变, 故走完各竖直段所需的总时间 t_g 就等于直接从 A 经 AB 到达 B 所需的时间 t_1 . 但走完各水平段所需的总时间则与质点在各水平段的速度大小有关, 即与各水平段的高度有关. 如果选择的路径, 使各水平段均位于三角形的底部 (即为 BC 边), 则质点在该水平段具有的速度最大, 所需时间应最短. 反之, 如果选择的路径, 使各水平段位于尽可能高的位置 (当然, 以不超出边线 AC 为条件), 则质点在各该水平段具有的速度最小, 所需时间应最长. 所以, 需时最长的路径为图1-1-2中虚线所示的锯齿状路径, 各锯齿的水平段 dx 和竖直段 dy 均为无穷小量, 各锯齿紧靠着 AC 而又都不超出 AC 边. 确定了需时最短和最长的路径, 求出相应的最短和最长时间, 即可得出两者之比.

【解】 1. 设 $AC = L$, 由运动学公式, 有

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9  -- EXAMPLE Ch1_Q1
10 -- Q: For right triangle ABC with hypotenuse AC and angle alpha between AC and horizontal BC,
11 -- a point mass moves from A to C starting from rest.
12 -- When moving along path AB then BC, times are t1 and t2;
13 -- when moving along AC, time is t3. Given t1 + t2 = t3, show that \sin \alpha + \frac{1}{2} \cos \alpha = 1.
14 theorem Ch1_Q1
15   (AC AB BC : Length)
16   (t1 t2 t3 : Time) (ht1':t1.val>0) (ht2':t2.val>0) (ht3':t3.val>0) (h':AC.val>0)
17   (\alpha : \mathbb{R}) (ha : \alpha \in Set.Ioo 0 (\pi/2))
18   (h_AB : AB = Real.sin \alpha * AC)
19   (h_BC : BC = Real.cos \alpha * AC)
20   (ht1 : t1 ** 2 = (2 * AB / g))
21   (ht2 : t2 ** 2 = ((Real.cot \alpha/2) * BC / g))
22   (ht3 : t3 ** 2 = (2 * AC / (Real.sin \alpha * g)))
23   (h_sum : t1 + t2 = t3) :
24   Real.sin \alpha + (1 / 2 : \mathbb{R}) * Real.cos \alpha = 1 := by sorry
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为了计算 t_2 , 考虑高度为 y 的任一水平段 dx , 质点以速度 v 经过 dx 所需时间为

$$dt_2 = \frac{dx}{v} = \frac{dx}{\sqrt{2g(L\sin\alpha - y)}}$$

因

$$\cot\alpha = -\frac{dx}{dy}$$

故

$$dt_2 = -\frac{\cot\alpha}{\sqrt{2g}} \cdot \frac{dy}{\sqrt{L\sin\alpha - y}}$$

所有水平段所需总时间为

$$\begin{aligned} t_2 &= \int dt_2 = -\frac{\cot\alpha}{\sqrt{2g}} \int_{L\sin\alpha}^0 \frac{dy}{\sqrt{L\sin\alpha - y}} \\ &= \cot\alpha \sqrt{\frac{2L\sin\alpha}{g}} = t_1 \cot\alpha = \frac{4}{3} t_1 \end{aligned}$$

所以

$$t_{\max} = t_1 + t_2 = \left(1 + \frac{4}{3}\right) t_1 = \frac{7}{3} t_1$$

$t_{\max}$ 与 $t_{\min}$ 之比为

$$\frac{t_{\max}}{t_{\min}} = \frac{7}{5}$$

【题2】 一质点以初速度 v_0 作直线运动, 所受阻力与其速度成正比. 试求当质点速度减为 $\frac{v_0}{n}$ ($n > 1$) 时, 质点经过的距离与质点所能行经的总距离之比.

【分析】 由题设, 质点的加速度 $a = \frac{dv}{dt} = -kv$, 积分可得出 $v(t)$, 再积分可得出 $x(t)$, 积分常数由初条件 $t=0$ 时 $v=v_0, x=0$ 确定. 上述 $v(t)$ 和 $x(t)$ 是 t 时刻质点的速度和位置, $x(t)$ 亦即经 t 时间后质点经过的距离. 总距离 x_{∞} 是 $t \rightarrow \infty$ 时的 x 值, 速度减为 $v_1 = \frac{v_0}{n}$ 时所经距离 $x(t_1)$ 则可由速度从 v_0 减为 v_1 所需的时间 t_1 求出.

【解】 质点沿直线运动, 取该直线为 x 坐标, 取原点 $x=0$ 为质点在 $t=0$ 时刻以初速度 v_0 开始运动的位置.

由题设, 在任意时刻 t , 质点的加速度为

$$a = \frac{dv}{dt} = -kv$$

式中 k 是常数. 分离变量, 得

$$\frac{dv}{v} = -k dt$$

积分, 得

$$\ln v = -kt + C$$

为了计算 t_2 , 考虑高度为 y 的任一水平段 dx , 质点以速度 v 经过 dx 所需时间为

$$dt_2 = \frac{dx}{v} = \frac{dx}{\sqrt{2g(L\sin\alpha - y)}}$$

因

$$\cot\alpha = -\frac{dx}{dy}$$

故

$$dt_2 = -\frac{\cot\alpha}{\sqrt{2g}} \cdot \frac{dy}{\sqrt{L\sin\alpha - y}}$$

所有水平段所需总时间为

$$\begin{aligned} t_2 &= \int dt_2 = -\frac{\cot\alpha}{\sqrt{2g}} \int_{L\sin\alpha}^0 \frac{dy}{\sqrt{L\sin\alpha - y}} \\ &= \cot\alpha \sqrt{\frac{2L\sin\alpha}{g}} = t_1 \cot\alpha = \frac{4}{3} t_1 \end{aligned}$$

所以

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$t_{\max}$ 与 $t_{\min}$ 之比为

$$\frac{t_{\max}}{t_{\min}} = \frac{7}{5}$$

[Problem 2] A particle moves in a straight line with an initial velocity v_0 . The resistance it encounters is proportional to its velocity. Determine the ratio of the distance traveled by the particle when its velocity reduces to v_0/n (where $n \geq 1$) to the total distance that the particle can travel.

【分析】 由题设, 质点的加速度 $a = \frac{dv}{dt} = -kv$, 积分可得出 $v(t)$, 再积分可得出 $x(t)$, 积分常数由初条件 $t=0$ 时 $v=v_0, x=0$ 确定. 上述 $v(t)$ 和 $x(t)$ 是 t 时刻质点的速度和位置, $x(t)$ 亦即经 t 时间后质点经过的距离. 总距离 x_{∞} 是 $t \rightarrow \infty$ 时的 x 值, 速度减为 $v_1 = \frac{v_0}{n}$ 时所经距离 $x(t_1)$ 则可由速度从 v_0 减为 v_1 所需的时间 t_1 求出.

【解】 质点沿直线运动, 取该直线为 x 坐标, 取原点 $x=0$ 为质点在 $t=0$ 时刻以初速度 v_0 开始运动的位置.

由题设, 在任意时刻 t , 质点的加速度为

$$a = \frac{dv}{dt} = -kv$$

式中 k 是常数. 分离变量, 得

$$\frac{dv}{v} = -k dt$$

积分, 得

$$\ln v = -kt + C$$

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26 -- EXAMPLE Ch1_Q2
27 -- Q: A particle moves along a straight line with initial velocity  $v_0$ .
28 -- It experiences resistance proportional to its velocity:  $a = dv/dt = -k v$ .
29 -- Let the particle's speed reduce to  $v_0 / n$  ( $n > 1$ ).
30 -- Show that the ratio of the distance traveled to the total distance is  $1 - 1/n$ .
31 theorem Ch1_Q2
32 {v : Time → Speed}
33 (hdif:Differentiable ℝ v)
34 (k : Scalar (-time_unit))
35 (v0 : Speed)
36 (n : ℝ)
37 (hk_pos : 0 < k.val)
38 (hn : n > 1)
39 (hv_diff : physderiv v=physmapcast (physmapsmul v (-k)) _ _ (by module) (by module))
40 (hv0 : v (0 • second) = v0)
41 (h_total : Tendsto
42   (fun T : ℝ => ∫ t in (0:ℝ)..T, (v {t}).val)
43   atTop (ℳ x_total))
44 (t1 : Time) (ht1 : v t1 = v0 / n)
45 : ∫ t in (0:ℝ)..t1.val, ((mathcast v) t) / x_total = 1 - 1 / n := by sorry

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$$\frac{v}{v_0} + \frac{x}{x_m} = 1$$

【题3】一质点以初速度 v_0 作直线运动,所受阻力与其速度的三次方成正比.试求质点速度和位置随时间的变化规律以及速度随位置的变化规律.

【分析】与上题相仿,从 $a = \frac{dv}{dt} = -kv^3$ 出发,通过积分可得 $v(t)$ 和 $x(t)$,消去 t 即可得出 $v(x)$, $v(x)$ 也可直接通过积分求得.

【解】取质点运动所循的直线为 x 轴,取坐标原点 $x=0$ 为质点在 $t=0$ 时刻以初速度 v_0 开始运动的位置.由题设,质点的加速度为

$$a = \frac{dv}{dt} = -kv^3$$

或

$$\frac{dv}{v^3} = -k dt$$

积分,得

$$\frac{1}{2v^2} = kt + C, \quad \text{或} \quad \frac{1}{v^2} = 2kt + C'$$

初条件为

$$t=0 \text{ 时}, \quad v = v_0$$

故积分常量

$$C' = \frac{1}{v_0^2}$$

代入上式,得

$$v(t) = v_0 \left(\frac{1}{1 + 2kv_0^2 t} \right)^{\frac{1}{2}} \quad (1)$$

因 $v = \frac{dx}{dt}$, 故

$$dx = v_0 \left(\frac{1}{1 + 2kv_0^2 t} \right)^{\frac{1}{2}} dt$$

积分,得

$$x = \frac{1}{kv_0} \sqrt{1 + 2kv_0^2 t} + C$$

初条件为

$$t=0 \text{ 时}, \quad x=0$$

故积分常量

$$C = -\frac{1}{kv_0}$$

代入上式,得

$$\frac{v}{v_0} + \frac{x}{x_m} = 1$$

[Problem 3] A particle moves in a straight line with an initial velocity v_0 . The resistance it experiences is proportional to the cube of its velocity. Please find the laws governing the changes of the particle's velocity and position over time, as well as the law of the velocity changing with position.

【分析】与上题相仿,从 $a = \frac{dv}{dt} = -kv^3$ 出发,通过积分可得 $v(t)$ 和 $x(t)$,消去 t 即可得出 $v(x)$, $v(x)$ 也可直接通过积分求得.

【解】取质点运动所循的直线为 x 轴,取坐标原点 $x=0$ 为质点在 $t=0$ 时刻以初速度 v_0 开始运动的位置.由题设,质点的加速度为

$$a = \frac{dv}{dt} = -kv^3$$

或

$$\frac{dv}{v^3} = -k dt$$

积分,得

$$\frac{1}{2v^2} = kt + C, \quad \text{或} \quad \frac{1}{v^2} = 2kt + C'$$

初条件为

$$t=0 \text{ 时}, \quad v = v_0$$

故积分常量

$$C' = \frac{1}{v_0^2}$$

代入上式,得

$$v(t) = v_0 \left(\frac{1}{1 + 2kv_0^2 t} \right)^{\frac{1}{2}} \quad (1)$$

因 $v = \frac{dx}{dt}$, 故

$$dx = v_0 \left(\frac{1}{1 + 2kv_0^2 t} \right)^{\frac{1}{2}} dt$$

积分,得

$$x = \frac{1}{kv_0} \sqrt{1 + 2kv_0^2 t} + C$$

初条件为

$$t=0 \text{ 时}, \quad x=0$$

故积分常量

$$C = -\frac{1}{kv_0}$$

代入上式,得

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47 -- EXAMPLE Ch1_Q3

48 -- Q: A particle moves along a straight line with initial velocity v_0 .

49 -- It experiences resistance proportional to the cube of its velocity: $a = dv/dt = -k v^3$.

50 -- Prove that its velocity as a function of time is $v(t) = v_0 * (1 / (1 + 2 k v_0^2 t))^{(1/2)}$.

51 theorem Ch1_Q3

52 (v0 v : Speed)

53 (vf : Time → Speed)

54 (hdif:Differentiable ℝ vf)

55 (k : Scalar (time_unit - 2 * length_unit))

56 (af : Speed → Acceleration)

57 (hdif1:Differentiable ℝ af)

58 (ha : af = fun v => -k * v ** 3)

59 (hv_diff : (physderiv vf) = (af ∘ vf))

60 (hv0 : vf (0 * second) = v0)

61 :

62 ∀ t : Time,

63 ((vf t) ** 2 * (1 * dimensionless + (2 : ℝ) * k * v0 ** 2 * t)).cast (by module) = v0 ** 2 := by sorry

64

$$x(t) = \frac{1}{kv_0} (\sqrt{1+2kv_0^2 t} - 1) \quad (2)$$

由(1)、(2)式,消去 t ,得

$$v(x) = \frac{v_0}{1+kv_0 x} \quad (3)$$

又,因

$$\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}$$

及

$$a = \frac{dv}{dt} = -kv^3$$

故

$$v \frac{dv}{dx} = -kv^3, \text{ 或 } \frac{dv}{v^3} = -k dx$$

积分,得

$$-\frac{1}{v} = -kx + C$$

初条件为

$$t=0 \text{ 时, } v=v_0, \quad x=0$$

故积分常量

$$C = -\frac{1}{v_0}$$

代入上式,得

$$\frac{1}{v} = kx + \frac{1}{v_0} = \frac{1+kv_0 x}{v_0}$$

或

$$v(x) = \frac{v_0}{1+kv_0 x}$$

此即上述(3)式.

【题4】 如图,在倾角为 θ 的山坡平面上有一门大炮,大炮相对于山坡的仰角为 α ,发射炮弹的初速为 v_0 . 试求炮弹着点的位置,并求能达到最大射程的仰角,忽略空气阻力.

【分析】 炮弹射出后,只受重力作用,水平方向的运动是匀速直线运动,竖直方向的运动是匀加速直线运动,由此不难求出炮弹的轨迹方程 $y(x)$,它和山坡的轨迹方程 $y = x \tan \theta$ (直线)联立求解得出的 (x_1, y_1) 就是炮弹着点的位置. 显然, x_1 (或 y_1) 将随仰角 α 变化,最大射程相应的仰角可由极值条件 $\frac{dx_1}{d\alpha} = 0$ (或 $\frac{dy_1}{d\alpha} = 0$) 求出. 也可由 x_1 (或 y_1) 的表达式给出射程的表达式,由后者得出最大射程相应的仰角.

【解】 如图,取直角坐标 Oxy , 原点 O 在大炮处, $t=0$ 时刻发射炮弹,于是炮弹的初始位置、初速度和加速度分别为

$$x(t) = \frac{1}{kv_0} (\sqrt{1+2kv_0^2 t} - 1) \quad (2)$$

由(1)、(2)式,消去 t ,得

$$v(x) = \frac{v_0}{1+kv_0 x} \quad (3)$$

又,因

$$\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}$$

及

$$a = \frac{dv}{dt} = -kv^3$$

故

$$v \frac{dv}{dx} = -kv^3, \text{ 或 } \frac{dv}{v^3} = -k dx$$

积分,得

$$-\frac{1}{v} = -kx + C$$

初条件为

$$t=0 \text{ 时, } v=v_0, \quad x=0$$

故积分常量

$$C = -\frac{1}{v_0}$$

代入上式,得

$$\frac{1}{v} = kx + \frac{1}{v_0} = \frac{1+kv_0 x}{v_0}$$

或

$$v(x) = \frac{v_0}{1+kv_0 x}$$

此即上述(3)式.

[Problem 4] As shown in the figure, on a sloping plane with an inclination angle of θ , there is a cannon. The elevation angle of the cannon relative to the slope is α , and the initial velocity of the fired projectile is v_0 . Determine the position of the impact point of the projectile and find the elevation angle that can achieve the maximum range, ignoring air resistance.

【分析】 炮弹射出后,只受重力作用,水平方向的运动是匀速直线运动,竖直方向的运动是匀加速直线运动,由此不难求出炮弹的轨迹方程 $y(x)$,它和山坡的轨迹方程 $y = x \tan \theta$ (直线)联立求解得出的 (x_1, y_1) 就是炮弹着点的位置. 显然, x_1 (或 y_1) 将随仰角 α 变化,最大射程相应的仰角可由极值条件 $\frac{dx_1}{d\alpha} = 0$ (或 $\frac{dy_1}{d\alpha} = 0$) 求出. 也可由 x_1 (或 y_1) 的表达式给出射程的表达式,由后者得出最大射程相应的仰角.

【解】 如图,取直角坐标 Oxy , 原点 O 在大炮处, $t=0$ 时刻发射炮弹,于是炮弹的初始位置、初速度和加速度分别为

```
65 -- EXAMPLE Ch1_Q4
66 -- Q: A cannon is placed on a slope inclined at an angle \(\theta\)\
67 -- and fires a projectile with initial speed \(\mathbf{v_0}\)\
68 -- at an elevation angle \(\alpha\)\ relative to the slope.
69 -- Prove that the elevation angle \(\alpha\)\ that
70 -- maximizes the range along the slope is \(\alpha = \frac{\pi}{4} - \frac{\theta}{2}\)\
71 theorem Ch1_Q4
72   (v0 : Speed)
73   (\theta \alpha : \mathbb{R})
74   (h0_range : 0 < \theta \wedge \theta < Real.pi / 2)
75   (vx vy : Speed)
76   (hvx : vx = cos \alpha * v0)
77   (hvy : vy = sin \alpha * v0)
78   (T:\mathbb{R} \rightarrow Time)
79   (hT:T=fun t=>(2*Real.sin t/Real.cos \theta) * v0/g)
80   (x:\mathbb{R} \rightarrow Length)
81   (hx:\forall t:\mathbb{R}, x t=(Real.cos t) * (v0*t)+(Real.sin \theta/2) * (g*(T t)**2)).cast (by module))
82   :
83   \forall \alpha \in Set.Icc (0 : \mathbb{R}) (Real.pi / 2 - \theta), x \alpha \leq x (\pi / 4 - \theta / 2) := by sorry
```

又,由(3)式得出射程 R 的公式为

$$\begin{aligned} R &= \frac{x_1}{\cos \theta} = \frac{v_0^2}{g \cos \theta} [\sin 2(\theta + \alpha) - 2 \tan \theta \cos^2(\theta + \alpha)] \\ &= \frac{2v_0^2}{g \cos \theta} [\tan(\theta + \alpha) - \tan \theta] \cos^2(\theta + \alpha) \\ &= \frac{2v_0^2}{g \cos \theta} \frac{\cos(\theta + \alpha) \sin \alpha}{\cos \theta} = \frac{v_0^2}{g \cos^2 \theta} [\sin(\theta + 2\alpha) - \sin \theta] \end{aligned}$$

当

$$\theta + 2\alpha = \frac{\pi}{2}, \text{即 } \alpha = \frac{\pi}{4} - \frac{\theta}{2}$$

时,达到最大射程,为

$$R_{\max} = \frac{v_0^2}{g \cos^2 \theta} (1 - \sin \theta)$$

【题5】 两质点在地面上同一地点以相同速率 v_0 从不同抛射角抛出。试证明,当两质点的射程 R 相同时,它们在空中飞行时间的乘积为 $\frac{2R}{g}$,忽略空气阻力。

【分析】 由质点斜抛运动的轨迹方程 $y(x)$ 及落地点 P 的纵坐标 $y=0$,联立求解可得出射程公式,进而得出飞行时间公式。显然,射程 R 和飞行时间 t 都会随初速率 v_0 以及抛射角 α 变化。现在,两质点的抛射角 α_1, α_2 不同,而射程相同均为 R ,这就要求 α_1 和 α_2 满足一定的关系。把这一关系代入飞行时间公式,当可证明 $t_1 t_2 = \frac{2R}{g}$ 。

【解】 取坐标如图,当质点从 O 点以初速率 v_0 和抛射角 α 抛出时,有

$$\begin{cases} x = v_0 \cos \alpha \cdot t \\ y = v_0 \sin \alpha \cdot t - \frac{1}{2} g t^2 \end{cases}$$

消去时间 t ,得出斜抛运动的轨迹方程为

$$y = x \tan \alpha - \frac{g}{2v_0^2 \cos^2 \alpha} x^2$$

落地点 P 的纵坐标为零,即

$$y = 0$$

由上两式联立解出的 x 即为射程 R ,

$$R = \frac{v_0^2 \sin 2\alpha}{g}$$

相应的飞行时间为

$$t = \frac{R}{v_0 \cos \alpha} = \frac{2v_0 \sin \alpha}{g}$$

现两质点从同一地点 O 以相同初速率 v_0 抛出,抛射角不同,分别为 α_1 和 α_2 ,但射程相同,故有

$$\sin 2\alpha_1 = \sin 2\alpha_2$$

• 10 •

又,由(3)式得出射程 R 的公式为

$$\begin{aligned} R &= \frac{x_1}{\cos \theta} = \frac{v_0^2}{g \cos \theta} [\sin 2(\theta + \alpha) - 2 \tan \theta \cos^2(\theta + \alpha)] \\ &= \frac{2v_0^2}{g \cos \theta} [\tan(\theta + \alpha) - \tan \theta] \cos^2(\theta + \alpha) \\ &= \frac{2v_0^2}{g \cos \theta} \frac{\cos(\theta + \alpha) \sin \alpha}{\cos \theta} = \frac{v_0^2}{g \cos^2 \theta} [\sin(\theta + 2\alpha) - \sin \theta] \end{aligned}$$

当

$$\theta + 2\alpha = \frac{\pi}{2}, \text{即 } \alpha = \frac{\pi}{4} - \frac{\theta}{2}$$

时,达到最大射程,为

$$R_{\max} = \frac{v_0^2}{g \cos^2 \theta} (1 - \sin \theta)$$

[Problem 5] Two particles are thrown from the same point on the ground with the same initial velocity v_0 at different launch angles. Prove that when the range R of the two particles is the same, the product of their flight times in the air is $2R/g$, assuming air resistance is negligible.

【分析】 由质点斜抛运动的轨迹方程 $y(x)$ 及落地点 P 的纵坐标 $y=0$,联立求解可得出射程公式,进而得出飞行时间公式。显然,射程 R 和飞行时间 t 都会随初速率 v_0 以及抛射角 α 变化。现在,两质点的抛射角 α_1, α_2 不同,而射程相同均为 R ,这就要求 α_1 和 α_2 满足一定的关系。把这一关系代入飞行时间公式,当可证明 $t_1 t_2 = \frac{2R}{g}$ 。

【解】 取坐标如图,当质点从 O 点以初速率 v_0 和抛射角 α 抛出时,有

$$\begin{cases} x = v_0 \cos \alpha \cdot t \\ y = v_0 \sin \alpha \cdot t - \frac{1}{2} g t^2 \end{cases}$$

消去时间 t ,得出斜抛运动的轨迹方程为

$$y = x \tan \alpha - \frac{g}{2v_0^2 \cos^2 \alpha} x^2$$

落地点 P 的纵坐标为零,即

$$y = 0$$

由上两式联立解出的 x 即为射程 R ,

$$R = \frac{v_0^2 \sin 2\alpha}{g}$$

相应的飞行时间为

$$t = \frac{R}{v_0 \cos \alpha} = \frac{2v_0 \sin \alpha}{g}$$

现两质点从同一地点 O 以相同初速率 v_0 抛出,抛射角不同,分别为 α_1 和 α_2 ,但射程相同,故有

$$\sin 2\alpha_1 = \sin 2\alpha_2$$

• 10 •

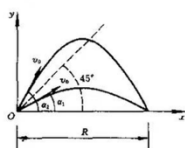


图 1-5-1

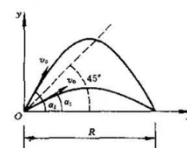


图 1-5-1

```
86 -- EXAMPLE Ch1_Q5
87 -- Q: Two particles are launched from the same point on level ground with the same speed $v_0$
88 -- but at different projection angles $\alpha_1, \alpha_2$.
89 -- Prove that if their horizontal ranges are equal (call it $R$),
90 -- then the product of their flight times satisfies $[t_1 t_2 = \frac{2R}{g}]$. \
91 theorem Ch1_Q5
92 (v0 : Speed)(hv0:v0.val>0)
93 (alpha_1 alpha_2 : ℝ)
94 (hdiff : alpha_1 ≠ alpha_2)
95 (halpha_1_range : 0 < alpha_1 ∧ alpha_1 < Real.pi / 2)
96 (halpha_2_range : 0 < alpha_2 ∧ alpha_2 < Real.pi / 2)
97 (t1 t2 : Time)
98 (R : Length)
99 (vx1 vy1 vx2 vy2 : Speed)
100 (hvx1 : vx1 = Real.cos alpha_1 * v0)
101 (hvy1 : vy1 = Real.sin alpha_1 * v0)
102 (hvx2 : vx2 = Real.cos alpha_2 * v0)
103 (hvy2 : vy2 = Real.sin alpha_2 * v0)
104 (ht1 : t1 = ((2:ℝ) * vy1) / g)
105 (ht2 : t2 = ((2:ℝ) * vy2) / g)
106 (hR1 : R = vx1 * t1)
107 (hR2 : R = vx2 * t2)
108 :
109 t1 * t2 = ((2:ℝ) * R) / g := by sorry
```

[例1] 如图2-1-1,在固定不动的圆柱体上绕有绳索,绳两端挂大、小两桶,其质量分别为 $M=1000\text{ kg}$ 和 $m=10\text{ kg}$,绳与圆柱体之间的摩擦系数为 $\mu=0.050$,绳的质量可以忽略.试问为使两桶静止不动,绳至少需绕多少圈.

[分析] 在大、小桶重量给定的条件下,为使两桶静止,要求绳的张力从大桶上端的 mg 沿桶面逐渐减少为小桶上端的 mg . 这表明,任一小段绳索两侧所受的张力,在靠近大桶的一侧稍大,在靠近小桶的一侧稍小.平衡是由绳与圆柱体之间的静摩擦力来维持的.在大、小桶重量给定的条件下,静摩擦力小所需绳索愈长,当绳与圆柱体之间为最大静摩擦力时所需绳索最短,即圈数最少.

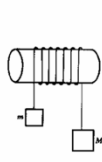


图2-1-1

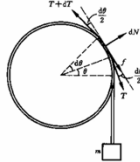


图2-1-2

[解] 如图2-1-2,隔取任意一小段绳子,它的长度为 $Rd\theta$ (R 是圆柱体的半径),它与小桶之间的绳索长度为 $R\theta$ (竖直下垂的那段绳索不计在内).它受到的作用力有,靠近小桶一侧的张力 T ,靠近大桶一侧的张力 $(T+dT)$,张力的方向如图2-1-2,移在作用点与圆柱体相切,支持力 dN 其方向沿径向,最大静摩擦力 $f=\mu dN$ 其方向与支持力垂直,因静止,合力为零,故有

$$\begin{cases} [(T+dT)+T]\sin\frac{\theta}{2}=dN \\ (T+dT)\cos\frac{\theta}{2}=T\cos\frac{\theta}{2}+\mu dN \end{cases}$$

即

[Problem 1] As in 2-1-1, a rope is wound around a stationary cylinder. The two ends of the rope are attached to two large and small buckets, with their masses being $M=1000\text{ kg}$ and $m=10\text{ kg}$ respectively. The friction coefficient between the rope and the cylinder is $\mu=0.050$. The mass of the rope can be neglected. Now, we need to determine how many times the rope must be wound to keep the two buckets stationary.

索逐渐减少为小桶上端的 mg . 这表明,任一小段绳索两侧所受的张力,在靠近大桶的一侧稍大,在靠近小桶的一侧稍小.平衡是由绳与圆柱体之间的静摩擦力来维持的.在大、小桶重量给定的条件下,静摩擦力小所需绳索愈长,当绳与圆柱体之间为最大静摩擦力时所需绳索最短,即圈数最少.

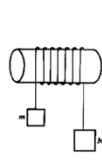


图2-1-1

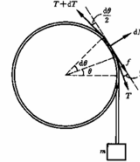


图2-1-2

[解] 如图2-1-2,隔取任意一小段绳子,它的长度为 $Rd\theta$ (R 是圆柱体的半径),它与小桶之间的绳索长度为 $R\theta$ (竖直下垂的那段绳索不计在内).它受到的作用力有,靠近小桶一侧的张力 T ,靠近大桶一侧的张力 $(T+dT)$,张力的方向如图2-1-2,移在作用点与圆柱体相切,支持力 dN 其方向沿径向,最大静摩擦力 $f=\mu dN$ 其方向与支持力垂直,因静止,合力为零,故有

$$\begin{cases} [(T+dT)+T]\sin\frac{\theta}{2}=dN \\ (T+dT)\cos\frac{\theta}{2}=T\cos\frac{\theta}{2}+\mu dN \end{cases}$$

即

```

9  -- EXAMPLE Ch2_Q1
10 -- Q: A uniform, massless rope is wrapped around a fixed cylinder;
11 -- its two ends hang vertically with a heavier bucket of mass \M=1000\ \mathrm{kg}\ on one end
12 -- and a lighter bucket of mass \m=10\ \mathrm{kg}\ on the other.
13 -- The coefficient of static friction between rope and cylinder is \(\mu=0.050\).
14 -- Let \(\theta\) be the number of full turns the rope makes around the cylinder.
15 -- Prove that the minimal number of turns required to keep the system in equilibrium
16 -- is \(\lceil \frac{1}{2\pi\mu} \ln \frac{M}{m} \rceil\).
17 theorem Ch2_Q1
18   (M m : Mass)
19   (\mu : R)
20   (n : R)
21   (\theta_total : R := 2 * Real.pi * n)
22   (T : R → Force)
23   (h_pos : 0 < M.val / m.val & 0 < m.val / M.val)
24   (hM_gt_m : M.val > m.val)
25   (T_light_def : T 0 = m * g)
26   (T_heavy_def : T \theta_total = M * g)
27   (capstan_differential : ∀ \theta : R, deriv (fun \theta' => (T \theta').val) \theta = \mu * (T \theta).val)
28   (capstan_integral : Real.log ((T \theta_total).val / (T 0).val) = \mu * \theta_total)
29   (\theta_def : \theta_total = 2 * Real.pi * n)
30   :
31   n = (1 / (2 * Real.pi * \mu)) * Real.log (M.val / m.val) := by sorry

```

College

Two point charges are located on the x-axis of a coordinate system: $q_1 = 1.0\text{ nC}$ is at $x = +2.0\text{ cm}$, and $q_2 = -3.0\text{ nC}$ is at $x = +4.0\text{ cm}$. What is the total electric force exerted by q_1 and q_2 on a charge $q_3 = 5.0\text{ nC}$ at $x = 0$?

```

theorem Electromagnetism_3_University
(hq1 : q1 = SI.nano (1 : coulomb))
(hq2 : q2 = SI.nano (-3 : coulomb))
(hq3 : q3 = SI.nano (5 : coulomb))
(hx1 : x1 = ((0.02:R) * meter))
(hx2 : x2 = ((0.04:R) * meter))
(hx3 : x3 = 0) (F : Force)
(hF : F = K * q3 * (q1 / (x1 - x3)^2 + q2 / (x2 - x3)^2)) :
F = ((9:R) * (10:R) ^ (-22:Q) / 32) * newton
:= by
simp [(←Scalar.val_inj), hF, hq1, hq2, hq3, hx1, hx2, hx3, K, SI.nano, coulomb, meter, newton]
norm_num

```

Competition-Easy

The plates of a parallel-plate capacitor are 2.50 mm apart, and each carries a charge of magnitude 80.0 nC . The plates are in vacuum. The electric field between the plates has a magnitude of $4.00 \times 10^6\text{ V/m}$. What is the capacitance?

```

theorem Ch13_electro_question_8
(V:Voltage) (C:Scalar)
(capacitance_unit) (q:Charge)
(E:Scalar)
(force_unit=charge_unit) (d:Length)
(h: capacitance_unit=charge_unit - voltage_unit)
(hC: C=(q/V).cast h) (hq:q=SI.nano (80 : coulomb))
(hV:V=E*d) (hE:E=(4e6:R) * StandardUnit _)
(hd:d=SI.milli (2.5:R) * meter) :
C=(1 / 125000000000:Q) * StandardUnit _
:= by
have hC_expanded : C = (q/(E*d)).cast h := by rw [hC, hV]
rw [hC_expanded, hq, hE, hd]
simp [nano, milli, coulomb, meter, ← Scalar.val_inj]
norm_num

```

Competition-Hard

Capstan law: If a rope of coefficient of friction μ wraps n turns ($\theta_{\text{total}} = 2\pi n$) around a post, the tension ratio between the heavy side M and the light side m satisfies $n = \frac{1}{2\pi\mu} \ln \frac{M}{m}$ assuming $M > m > 0$ and $\mu > 0$.

```

theorem Ch2_Q1
(M m : Mass)
(\mu : R)
(n : R)
(\theta_total : R := 2 * Real.pi * n)
(T : R → Force)
(h_pos : 0 < M.val / m.val & 0 < m.val / M.val)
(hM_gt_m : M.val > m.val)
(T_light_def : T 0 = m * g)
(T_heavy_def : T \theta_total = M * g)
(capstan_differential : ∀ \theta : R, deriv (fun \theta' => (T \theta').val) \theta = \mu * (T \theta).val)
(capstan_integral : Real.log ((T \theta_total).val / (T 0).val) = \mu * \theta_total)
(\theta_def : \theta_total = 2 * Real.pi * n) :
n = (1 / (2 * Real.pi * \mu)) * Real.log (M.val / m.val) := by
rcases h_pos with (hM, hm, hmu)
have h1 : Real.log ((T \theta_total).val / (T 0).val) =
Real.log ((M * g).val / (m * g).val) := by rw [T_heavy_def, T_light_def]
have h2 : Real.log ((M * g).val / (m * g).val) = Real.log (M.val / m.val) := by
have h3 : (M * g).val / (m * g).val = M.val / m.val := by
field_simp; ring_nf; simp; ring
rw [h3]
have h3 : Real.log (M.val / m.val) = \mu * \theta_total := by linarith
[capstan_integral, h1, h2]
have h4 : Real.log (M.val / m.val) = \mu * (2 * Real.pi * n) := by rw [theta_def] at h3; linarith
have h5 : \mu \ne 0 := by linarith
have h6 : Real.pi \ne 0 := Real.pi_ne_zero
have h7 : Real.log (M.val / m.val) = 2 * Real.pi * \mu * n := by linarith [h4]
have h8 : n = (Real.log (M.val / m.val)) / (2 * Real.pi * \mu) := by field_simp; linarith
rw [h8]; field_simp; ring_nf; field_simp; ring

```

Figure 5: Three sampled physics questions from *College Textbook*, *Olympics-Easy*, *Olympics-Hard* problems. Each example shows the natural language problem statement followed by its corresponding Lean formalization with a verified proof.

$$\begin{cases} (2T + dT) \sin \frac{d\theta}{2} = dN \\ dT \cos \frac{d\theta}{2} = \mu dN \end{cases}$$

因 $d\theta$ 很小, 有

$$\sin \frac{d\theta}{2} \approx \frac{d\theta}{2}, \quad \cos \frac{d\theta}{2} \approx 1$$

故为

$$\begin{cases} 2T \frac{d\theta}{2} + dT \frac{d\theta}{2} = dN \\ dT = \mu dN \end{cases}$$

忽略高级小量, 得

$$\begin{cases} T d\theta = dN \\ dT = \mu dN \end{cases}$$

消去 dN , 得

$$\frac{dT}{T} = \mu d\theta$$

积分, 得

$$\ln T = \mu \theta + C$$

因 $\theta = 0$ 时, $T = mg$, 故积分常量为

$$C = \ln mg$$

代入, 得

$$T = mg e^{\mu \theta}$$

这就是为使两桶静止, 绳索中张力随 θ 的变化. 与小桶连接处 ($\theta = 0$), 绳的张力最小为 $T_{\min} = mg$; 与大桶连接处, 绳的张力最大应为 $T_{\max} = Mg$. 设绳索共绕 n 圈, 由上式, 得

$$T_{\max} = T_{\min} e^{\mu \cdot 2\pi n}$$

即

$$Mg = mg e^{\mu \cdot 2\pi n}$$

故

$$n = \frac{1}{2\pi\mu} \ln \frac{M}{m} = \frac{\ln 100}{0.1\pi} = 14.7 \text{ 圈} \approx 15 \text{ 圈}$$

【题2】 如图 2-2-1, 细杆一端支在地面上, 以恒定的角速度 ω 绕通过支点的竖直轴旋转, 杆与地面的夹角为 α , 质量为 m 的小环套在杆上, 可以沿杆滑动, 环与杆之间的摩擦系数为 μ . 试问小环处于什么位置上能维持稳定运动.

【分析】 如图 2-2-2, 小环受到的作用力有, 重力 mg , 杆的支持力 N , 以及摩擦力 f . 小环能在杆上稳定地以角速度 ω 作圆周运动, 既不上滑也不下滑, 要求上述三个力的合力的竖直分量为零, 水平分量提供所需之向心力.

$$\begin{cases} (2T + dT) \sin \frac{d\theta}{2} = dN \\ dT \cos \frac{d\theta}{2} = \mu dN \end{cases}$$

因 $d\theta$ 很小, 有

$$\sin \frac{d\theta}{2} \approx \frac{d\theta}{2}, \quad \cos \frac{d\theta}{2} \approx 1$$

故为

$$\begin{cases} 2T \frac{d\theta}{2} + dT \frac{d\theta}{2} = dN \\ dT = \mu dN \end{cases}$$

忽略高级小量, 得

$$\begin{cases} T d\theta = dN \\ dT = \mu dN \end{cases}$$

消去 dN , 得

$$\frac{dT}{T} = \mu d\theta$$

积分, 得

$$\ln T = \mu \theta + C$$

因 $\theta = 0$ 时, $T = mg$, 故积分常量为

$$C = \ln mg$$

代入, 得

$$T = mg e^{\mu \theta}$$

这就是为使两桶静止, 绳索中张力随 θ 的变化. 与小桶连接处 ($\theta = 0$), 绳的张力最小为 $T_{\min} = mg$; 与大桶连接处, 绳的张力最大应为 $T_{\max} = Mg$. 设绳索共绕 n 圈, 由上式, 得

$$T_{\max} = T_{\min} e^{\mu \cdot 2\pi n}$$

即

$$Mg = mg e^{\mu \cdot 2\pi n}$$

故

$$n = \frac{1}{2\pi\mu} \ln \frac{M}{m} = \frac{\ln 100}{0.1\pi} = 14.7 \text{ 圈} \approx 15 \text{ 圈}$$

[Problem 2] As in 2-2-1, the end of the thin rod is supported on the ground, rotating around a vertical axis passing through the support point with a constant angular velocity ω . The angle between the rod and the ground is α , and the mass of the small ring is m . The ring is placed on the rod and can slide along it. The friction coefficient between the ring and the rod is μ . What position should the small ring be in to maintain a stable motion?

【分析】 如图 2-2-2, 小环受到的作用力有, 重力 mg , 杆的支持力 N , 以及摩擦力 f . 小环能在杆上稳定地以角速度 ω 作圆周运动, 既不上滑也不下滑, 要求上述三个力的合力的竖直分量为零, 水平分量提供所需之向心力.

```

29 -- EXAMPLE Ch2_Q2
30 -- Q: A thin rod rotates about a vertical axis passing through its support with constant angular velocity \(\omega\).
31 -- The rod makes an angle \(\alpha\) with the horizontal.
32 -- A small ring of mass \((m)\) can slide along the rod,
33 -- and the coefficient of friction between the ring and the rod is \(\mu\).
34 -- Prove that the stable positions \((L_1)\) and \((L_2)\) of the ring,
35 -- measured along the rod from the axis of rotation, satisfy
36 -- \[ L_1 = \frac{g(\sin \alpha - \mu \cos \alpha)}{\omega^2 \cos \alpha (\cos \alpha + \mu \sin \alpha)}, \]
37 -- \[ L_2 = \frac{g(\sin \alpha + \mu \cos \alpha)}{\omega^2 \cos \alpha (\cos \alpha - \mu \sin \alpha)}. \]
38 theorem Ch2_Q2
39   (m : Mass)
40   (L : Length)
41   (\alpha : ℝ)
42   (\omega : Scalar (-time_unit))
43   (\mu : ℝ)
44   (N f : Force)
45   (h_pos : 0 < m.val ∧ 0 < μ ∧ 0 < ω ∧ 0 < g)
46   (h_cos : Real.cos α ≠ 0)
47   (h_horiz : (Real.sin α * N) - (Real.cos α * f) = (ω ** 2) * m * (Real.cos α * L))
48   (h_vert : (Real.cos α * N) + (Real.sin α * f) = g * m)
49   (f_def : f = Real.sin α * m * g - Real.cos α ^ 2 * m * ω ** 2 * L / 1)
50   (N_def : N = Real.cos α * m * g + (2 * Real.sin α * Real.cos α) * m * ω ** 2 * L / 1)
51   (fric_bound : ||f.val|| ≤ μ * ||N.val||)
52   :
53   ∀ (ε : ℤ), (ε = 1 ∨ ε = -1) →
54     (f = ((ε : ℝ) * μ * N) →
55       L = ((Real.sin α - (ε : ℝ) * μ * Real.cos α) * g) /
56         (Real.cos α * (Real.cos α + (ε : ℝ) * μ * Real.sin α)) * ω ** 2) := by sorry

```

$$L_1 = \frac{g(\sin \alpha - \mu \cos \alpha)}{\omega^2 \cos \alpha (\cos \alpha + \mu \sin \alpha)}$$
 当 $f = -\mu N$ 时, $L = L_2$, 由(3)、(4)式, 并将 L_0 代入, 得

$$mg \sin \alpha - m\omega^2 L_2 \cos^2 \alpha = -\mu mg \cos \alpha - \mu m\omega^2 L_2 \sin \alpha \cos \alpha$$
 故

$$L_2 = \frac{g(\sin \alpha + \mu \cos \alpha)}{\omega^2 \cos \alpha (\cos \alpha - \mu \sin \alpha)}$$
 小环随杆稳定运动的范围是

$$\Delta L = L_2 - L_1 = \frac{2\mu g}{\omega^2 \cos^3 \alpha (1 - \mu^2 \tan^2 \alpha)}$$
 以上结果可用如图 2-2-3 所示的 $f(L)$ 曲线表示. 在小环稳定运动的 $L_1 < L < L_2$ 范围内, 由(3)式可知, 随着 L 的增大, f 线性地减小. 在 L_0 处, N 的竖直分量刚好与 mg 抵消, N 的水平分量刚好提供小环匀速圆周运动所需的向心力, 摩擦力 f 为零. 在 $L_1 < L < L_0$ 范围, 随着 L 从 L_0 减小, N 减小 (由(4)式), 其竖直分量不足以抵消 mg , 小环有下滑趋势, 导致方向沿杆向上不为零的摩擦力 f , N 与 f 的竖直分量之和与 mg 抵消, N 与 f 的水平分量之和提供环圆周运动的向心力, 环稳定运动, 在 $L = L_1$ 处, $f = \mu N$ 达到最大静摩擦力, 所以, 在 $L_1 < L < L_0$ 范围内环稳定运动. 当 $L < L_1$, 随着 L 减小, N 减小, 最大静摩擦力 $f = \mu N$ (方向沿杆向上) 减小, N 与 f 的竖直分量不足以抵消 mg , 环向下滑动, 无法稳定运动.

与此类似, 在 $L_0 < L < L_2$ 范围内, 环稳定运动, 在 L_2 处达到最大静摩擦力 $f = \mu N$ (方向沿杆向下). 在 $L > L_2$ 范围内, 环向上滑动, 无法稳定运动.

【题 3】 如图, 在半径为 R 的空心球壳内壁, 有一可当作质点的小球沿固定的水平圆周作匀速运动. 小球与空心球壳球心的连线与铅垂线的夹角为 θ , 小球与空心球壳内壁之间的摩擦系数为 μ . 试求小球能稳定运动的速度范围.

【分析】 与上题相仿, 也是讨论能维持稳定运动的问题. 区别在于, 上题小环随杆旋转, 可以沿杆上、下滑动, 求的是能够维持稳定运动的空间范围; 本题小球运动的位置限定, 求的是能够维持稳定运动的速度范围.

如图, 小球所受作用力仍为重力、支持力和摩擦力, 它们是维持小球稳定运动的依据. 关键仍在于, 摩擦力 f 的大小可在零和最大静摩擦力 μN 之间, 摩擦力的方向可沿球壳切线向上或向下. 换言之, 阻止小球下滑或上滑的, 沿球壳内壁切线方向向上或向下的最大静摩擦力, 确定了小球能在给定位置稳定运动的速度范围.

【解】 如图, 小球受重力 mg , 支持力 N , 摩擦力 (假设方向沿切线向上), 小球的运动方程及有关

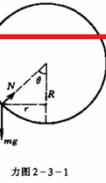


图 2-3-1

$$L_1 = \frac{g(\sin \alpha - \mu \cos \alpha)}{\omega^2 \cos \alpha (\cos \alpha + \mu \sin \alpha)}$$
 当 $f = -\mu N$ 时, $L = L_2$, 由(3)、(4)式, 并将 L_0 代入, 得

$$mg \sin \alpha - m\omega^2 L_2 \cos^2 \alpha = -\mu mg \cos \alpha - \mu m\omega^2 L_2 \sin \alpha \cos \alpha$$
 故

$$L_2 = \frac{g(\sin \alpha + \mu \cos \alpha)}{\omega^2 \cos \alpha (\cos \alpha - \mu \sin \alpha)}$$
 小环随杆稳定运动的范围是

$$\Delta L = L_2 - L_1 = \frac{2\mu g}{\omega^2 \cos^3 \alpha (1 - \mu^2 \tan^2 \alpha)}$$
 以上结果可用如图 2-2-3 所示的 $f(L)$ 曲线表示. 在小环稳定运动的 $L_1 < L < L_2$ 范围内, 由(3)式可知, 随着 L 的增大, f 线性地减小. 在 L_0 处, N 的竖直分量刚好与 mg 抵消, N 的水平分量刚好提供小环匀速圆周运动所需的向心力, 摩擦力 f 为零. 在 $L_1 < L < L_0$ 范围, 随着 L 从 L_0 减小, N 减小 (由(4)式), 其竖直分量不足以抵消 mg , 小环有下滑趋势, 导致方向沿杆向上不为零的摩擦力 f , N 与 f 的竖直分量之和与 mg 抵消, N 与 f 的水平分量之和提供环圆周运动的向心力, 环稳定运动, 在 $L = L_1$ 处, $f = \mu N$ 达到最大静摩擦力, 所以, 在 $L_1 < L < L_0$ 范围内环稳定运动. 当 $L < L_1$, 随着 L 减小, N 减小, 最大静摩擦力 $f = \mu N$ (方向沿杆向上) 减小, N 与 f 的竖直分量不足以抵消 mg , 环向下滑动, 无法稳定运动.

与此类似, 在 $L_0 < L < L_2$ 范围内, 环稳定运动, 在 L_2 处达到最大静摩擦力 $f = \mu N$ (方向沿杆向下). 在 $L > L_2$ 范围内, 环向上滑动, 无法稳定运动.

【分析】 与上题相仿, 也是讨论能维持稳定运动的问题. 区别在于, 上题小环随杆旋转, 可以沿杆上、下滑动, 求的是能够维持稳定运动的空间范围; 本题小球运动的位置限定, 求的是能够维持稳定运动的速度范围.

如图, 小球所受作用力仍为重力、支持力和摩擦力, 它们是维持小球稳定运动的依据. 关键仍在于, 摩擦力 f 的大小可在零和最大静摩擦力 μN 之间, 摩擦力的方向可沿球壳切线向上或向下. 换言之, 阻止小球下滑或上滑的, 沿球壳内壁切线方向向上或向下的最大静摩擦力, 确定了小球能在给定位置稳定运动的速度范围.

【解】 如图, 小球受重力 mg , 支持力 N , 摩擦力 (假设方向沿切线向上), 小球的运动方程及有关

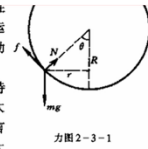


图 2-3-1

```

63 -- EXAMPLE Ch2_Q3
64 -- Q: A small ball of mass \m moves inside a hollow sphere of radius \R,
65 -- constrained to a horizontal circular path.
66 -- The angle between the line connecting the ball to the sphere's center and the vertical is \theta.
67 -- The coefficient of static friction between the ball and the inner surface of the sphere is \mu.
68 -- Prove that the range of velocities \v for which the ball can move stably is
69 -- \[ v_{\min} = \sqrt{\frac{g(\sin \theta - \mu \cos \theta) R \sin \theta}{\cos \theta + \mu \sin \theta}}, \]
70 -- \[ v_{\max} = \sqrt{\frac{g(\sin \theta + \mu \cos \theta) R \sin \theta}{\cos \theta - \mu \sin \theta}}. \]
71 theorem Ch2_Q3
72 (m : Mass) (R : Length) (theta : Real) (v : Speed) (mu : Real) (N f : Force)
73 (h_pos : 0 < m.val ^ 0 < mu ^ 0 < g.val)
74 (h_sin_cos : Real.sin theta != 0 ^ Real.cos theta != 0)
75 (r_def : r = Real.sin theta * R)
76 (h_horiz : Real.sin theta * N - Real.cos theta * f = m * v**2 / r)
77 (h_vert : Real.cos theta * N + Real.sin theta * f = m * g)
78 (f_def : f = m * (Real.sin theta * g - Real.cos theta * v**2 / r / 1))
79 (N_def : N = m * (Real.cos theta * g + Real.sin theta * v**2 / r / 1))
80 (fric_bound : ||f.val|| <= mu * ||N.val||)
81 :
82 V (epsilon : Z), (epsilon = 1 ^ epsilon = -1) ->
83 (f = (epsilon : Real) * mu * N ->
84 v**2 = ((Real.sin theta - (epsilon : Real) * mu * Real.cos theta) * g * (Real.sin theta * R) /
85 (Real.cos theta + (epsilon : Real) * mu * Real.sin theta))) := by sorry
  
```

College Level

```

theorem University_Mechanics_3
(x_0 x_1 : Length)
(t_0 t_1 dt : Time)
(v_0 v_1 : Speed)
(a : Acceleration)
(xf xfl : Time -> Length) (vf : Time -> Speed)
(ht0 : t_0 = 0 * second)
(ht1 : t_1 = 4 * second)
(ht : dt = t_1 - t_0)
(ha : a = (v_1 - v_0) / dt)
(hv : V t, vf t = v_0 + a * t / 1)
(hv1 : v_1 = vf dt)
(hx : V t, xf t = (a * t**2) / 2 + v_0 * t)
(hxx : V t, xfl t = (3 * meter / second**2) * t**2 -
2 * meter / second * t)
(hxxx : xf = xfl)
(a = 6 * meter / second**2 ^
v_0 = -2 * meter / second)
:= by sorry
  
```

Competition Level

```

theorem competition_mechanics_Ch2_Q32
(m : Mass) (R : Length) (theta : Real) (v : Speed) (mu : Real) (N f : Force)
(h_pos : 0 < m.val ^ 0 < mu ^ 0 < g.val)
(h_sin_cos : Real.sin theta != 0 ^ Real.cos theta != 0)
(r_def : r = Real.sin theta * R)
(h_horiz : Real.sin theta * N - Real.cos theta * f =
m * v**2 / r)
(h_vert : Real.cos theta * N + Real.sin theta * f = m * g)
(f_def : f = m * (Real.sin theta * g -
Real.cos theta * v**2 / r / 1))
(N_def : N = m * (Real.cos theta * g +
Real.sin theta * v**2 / r / 1))
(fric_bound : ||f.val|| <= mu * ||N.val||)
V (epsilon : Z), (epsilon = 1 ^ epsilon = -1) ->
f = (epsilon : Real) * mu * N -> v**2 =
((Real.sin theta - (epsilon : Real) * mu * Real.cos theta) * g *
(Real.sin theta * R) /
(Real.cos theta + (epsilon : Real) * mu * Real.sin theta)))
:= by sorry
  
```

Figure 3: Two examples from *LeanPhysBench* demonstrating different difficulty levels.

消去 N , 得

$$f = m(g \sin \theta - \frac{v^2}{r} \cos \theta)$$

消去 f , 得

$$N = m(g \cos \theta + \frac{v^2}{r} \sin \theta)$$

由 f 的表达式可知:

1. 当 $g \sin \theta = \frac{v^2}{r} \cos \theta$, 即当 $v = \sin \theta \sqrt{\frac{gR}{\cos \theta}}$ 时,
 $f = 0$
2. 当 $g \sin \theta > \frac{v^2}{r} \cos \theta$, 即当 $v < \sin \theta \sqrt{\frac{gR}{\cos \theta}}$ 时,
 $f > 0$, 方向向上
3. 当 $g \sin \theta < \frac{v^2}{r} \cos \theta$, 即当 $v > \sin \theta \sqrt{\frac{gR}{\cos \theta}}$ 时,
 $f < 0$, 方向向下

当 $f > 0$ 时, 若 v 减小, 则所需之摩擦力增大. 设当 $v = v_{\min}$ 时, $f = f_{\max} = \mu N$, 则由 f 和 N 的表达式, 有

$$\mu m(g \cos \theta + \frac{v_{\min}^2}{r} \sin \theta) = m(g \sin \theta - \frac{v_{\min}^2}{r} \cos \theta)$$

解出 $v_{\min}$ 为

$$v_{\min} = \sqrt{\frac{g(\sin \theta - \mu \cos \theta)r}{\cos \theta + \mu \sin \theta}} = \sqrt{\frac{g(\sin \theta - \mu \cos \theta)R \sin \theta}{\cos \theta + \mu \sin \theta}}$$

当 $f < 0$ 时, 若 v 增大, 则所需之摩擦力的绝对值增大. 设当 $v = v_{\max}$ 时, $|f| = |f|_{\max} = \mu N$, 则由 f 和 N 的表达式, 有

$$-\mu m(g \cos \theta + \frac{v_{\max}^2}{r} \sin \theta) = m(g \sin \theta - \frac{v_{\max}^2}{r} \cos \theta)$$

解出 $v_{\max}$ 为

$$v_{\max} = \sqrt{\frac{g(\sin \theta + \mu \cos \theta)R \sin \theta}{\cos \theta - \mu \sin \theta}}$$

小球能在空心球壳内壁 $r = R \sin \theta$ 处稳定运动的速度范围是

$$v_{\min} \leq v \leq v_{\max}$$

【题4】 如图2-4-1, 在地面上有一倾角为 θ , 质量为 M 的斜面体, 斜面体上有一质量为 m

的木块. 设地面与斜面体之间以及斜面体与木块之间均光滑无摩擦. 试求 M 与 m 相对于地面的加速度 a_M 与 a_m , 以及木块 m 所受的支持力.

消去 N , 得

$$f = m(g \sin \theta - \frac{v^2}{r} \cos \theta)$$

消去 f , 得

$$N = m(g \cos \theta + \frac{v^2}{r} \sin \theta)$$

由 f 的表达式可知:

1. 当 $g \sin \theta = \frac{v^2}{r} \cos \theta$, 即当 $v = \sin \theta \sqrt{\frac{gR}{\cos \theta}}$ 时,
 $f = 0$
2. 当 $g \sin \theta > \frac{v^2}{r} \cos \theta$, 即当 $v < \sin \theta \sqrt{\frac{gR}{\cos \theta}}$ 时,
 $f > 0$, 方向向上
3. 当 $g \sin \theta < \frac{v^2}{r} \cos \theta$, 即当 $v > \sin \theta \sqrt{\frac{gR}{\cos \theta}}$ 时,
 $f < 0$, 方向向下

当 $f > 0$ 时, 若 v 减小, 则所需之摩擦力增大. 设当 $v = v_{\min}$ 时, $f = f_{\max} = \mu N$, 则由 f 和 N 的表达式, 有

$$\mu m(g \cos \theta + \frac{v_{\min}^2}{r} \sin \theta) = m(g \sin \theta - \frac{v_{\min}^2}{r} \cos \theta)$$

解出 $v_{\min}$ 为

$$v_{\min} = \sqrt{\frac{g(\sin \theta - \mu \cos \theta)r}{\cos \theta + \mu \sin \theta}} = \sqrt{\frac{g(\sin \theta - \mu \cos \theta)R \sin \theta}{\cos \theta + \mu \sin \theta}}$$

当 $f < 0$ 时, 若 v 增大, 则所需之摩擦力的绝对值增大. 设当 $v = v_{\max}$ 时, $|f| = |f|_{\max} = \mu N$, 则由 f 和 N 的表达式, 有

$$-\mu m(g \cos \theta + \frac{v_{\max}^2}{r} \sin \theta) = m(g \sin \theta - \frac{v_{\max}^2}{r} \cos \theta)$$

解出 $v_{\max}$ 为

$$v_{\max} = \sqrt{\frac{g(\sin \theta + \mu \cos \theta)R \sin \theta}{\cos \theta - \mu \sin \theta}}$$

小球能在空心球壳内壁 $r = R \sin \theta$ 处稳定运动的速度范围是

$$v_{\min} \leq v \leq v_{\max}$$

[Problem 4 As in Figure 2-4-1, there is an inclined plane with an angle of θ and a mass of M on the ground. There is a wooden block of mass m on the inclined plane. Assuming that the ground and the inclined plane, as well as the inclined plane and the wooden block, are all frictionless and smooth, we need to find the accelerations a_M and a_m of M and m relative to the ground, as well as the supporting force exerted on the wooden block m .

```

78 -- EXAMPLE Ch2_Q4
79 -- Q: A block of mass $m$ rests on a frictionless inclined plane of mass $M$ and angle $\theta$.
80 -- Both the block and the plane are free to move on a frictionless horizontal surface.
81 -- Prove that the accelerations of the block $(a_x, a_y)$ and
82 -- the plane $a_M$ with respect to the ground are $[ a_x = -\frac{M \sin \theta \cos \theta}{M + m \sin^2 \theta} g, \ ]$
83 -- $[ a_y = -\frac{(m+M) \sin^2 \theta}{M + m \sin^2 \theta} g, \ ]$
84 -- $[ a_M = \frac{m \sin \theta \cos \theta}{M + m \sin^2 \theta} g. \ ]$
85 theorem Ch2_Q4
86   (m M : Mass)
87   (θ : ℝ)
88   (a_x a_y a_M : Acceleration)
89   (N : Force)
90   (h_pos : 0 < m.val ∧ 0 < M.val ∧ 0 < g)
91   (h_sin : Real.sin θ ≠ 0)
92   (h_cos : Real.cos θ ≠ 0)
93   (h_block_x : m * a_x = - Real.sin θ * N)
94   (h_block_y : m * a_y = Real.cos θ * N - m * g)
95   (h_plane : M * a_M = Real.sin θ * N)
96   (h_constraint : Real.sin θ * a_x - Real.cos θ * a_y = Real.sin θ * a_M)
97   :
98   (a_x, a_y, a_M, N) =
99     ( - ((Real.sin θ * Real.cos θ) * M) / (M + Real.sin θ ^2 * m) * g,
100       - (Real.sin θ ^2 * (m + M)) / (M + Real.sin θ ^2 * m) * g,
101       ((Real.sin θ * Real.cos θ) * m) / (M + Real.sin θ ^2 * m) * g,
102       (Real.cos θ * m * M) / (M + Real.sin θ ^2 * m) * g ) := by sorry

```

于是

$$\tan \theta = \frac{a_y}{a_x} = \frac{a_y}{a_x - a_M}$$

即

$$a_x \sin \theta - a_y \cos \theta = a_M \sin \theta. \quad (4)$$

(1)、(2)、(3)、(4)式联立求解,得

$$\begin{aligned} a_x &= -\frac{M \sin \theta \cos \theta}{M + m \sin^2 \theta} g \\ a_y &= -\frac{(m + M) \sin^2 \theta}{M + m \sin^2 \theta} g \\ a_M &= \frac{m \sin \theta \cos \theta}{M + m \sin^2 \theta} g \\ N &= \frac{m M \cos \theta}{M + m \sin^2 \theta} g \end{aligned}$$

式中负号表明了木块加速度的实际方向。当 $m \ll M$ 时, $a_M \approx 0$, M 几乎不动, 上述结果简化为斜面体固定不动的结果。

【题5】 如图2-5-1, 一个半径为 $R = 0.5 \text{ m}$ 的空心球壳绕本身的竖直直径旋转, 角速度为 $\omega = 5 \text{ s}^{-1}$, 在空心球壳内高度为 $\frac{R}{2}$ 处有一小木块(可当作质点)同球壳一起旋转。

试求: 1. 摩擦系数至少是多少才能实现这一情况。2. 当 $\omega = 8 \text{ s}^{-1}$ 时, 实现这一情况的条件是什么。3. 研究以下两种情形运动的稳定性: (a) 木块位置有微小变动; (b) 空心球壳的角速度有微小变动。

【分析】 如图2-5-2, 小木块受重力 mg 、支持力 N 、摩擦力 f , 合力的水平分量提供木块作圆周运动的向心力, 合力的竖直分量应等于零。小木块与球无相对滑动, 但有相对滑动趋势, 故 f 为静摩擦力, 取值为 $0 \leq f \leq \mu N$ 。因题目求的是“最小”的摩擦系数 μ , 故 f 应取最大静摩擦力, 即 $f = \mu N$ 。

当 ω 不同时, 所需的最小的 μ 不同。另外, 当木块在不同 θ 处时, 所需的最小的 μ 也不同。因此, 木块在不同的 θ 处, 以不同的 ω 随球旋转时, 所需的最小的 μ 是 θ 和 ω 的函数, 即 $\mu =$

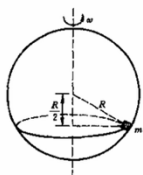


图2-5-1

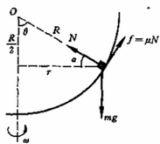


图2-5-2

于是

$$\tan \theta = \frac{a_y}{a_x} = \frac{a_y}{a_x - a_M}$$

即

$$a_x \sin \theta - a_y \cos \theta = a_M \sin \theta. \quad (4)$$

(1)、(2)、(3)、(4)式联立求解,得

$$\begin{aligned} a_x &= -\frac{M \sin \theta \cos \theta}{M + m \sin^2 \theta} g \\ a_y &= -\frac{(m + M) \sin^2 \theta}{M + m \sin^2 \theta} g \\ a_M &= \frac{m \sin \theta \cos \theta}{M + m \sin^2 \theta} g \\ N &= \frac{m M \cos \theta}{M + m \sin^2 \theta} g \end{aligned}$$

式中负号表明了木块加速度的实际方向。当 $m \ll M$ 时, $a_M \approx 0$, M 几乎不动, 上述结果简化为斜面体固定不动的结果。

[Problem 5] As shown in Figure 2-5-1, a hollow spherical shell of radius $R = 0.5 \text{ m}$ rotates around its vertical axis. The angular velocity is $\omega = 5 \text{ rad/s}$. There is a small wooden block (treated as a particle) at a height of h from the center inside the hollow spherical shell, which rotates together with the shell.

Please calculate:

1. What is the minimum friction coefficient required to achieve this situation.

2. What are the conditions for achieving this situation when $\omega = 8 \text{ rad/s}$.

3. Study the stability of the following two scenarios of motion:

(a) There is a slight change in the position of the wooden block;

(b) The angular velocity of the hollow spherical shell has a slight change.

【分析】 如图2-5-2, 小木块受重力 mg 、支持力 N 、摩擦力 f , 合力的水平分量提供木块作圆周运动的向心力, 合力的竖直分量应等于零。小木块与球无相对滑动, 但有相对滑动趋势, 故 f 为静摩擦力, 取值为 $0 \leq f \leq \mu N$ 。因题目求的是“最小”的摩擦系数 μ , 故 f 应取最大静摩擦力, 即 $f = \mu N$ 。

当 ω 不同时, 所需的最小的 μ 不同。另外, 当木块在不同 θ 处时, 所需的最小的 μ 也不同。因此, 木块在不同的 θ 处, 以不同的 ω 随球旋转时, 所需的最小的 μ 是 θ 和 ω 的函数, 即 $\mu =$

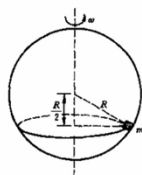


图2-5-1

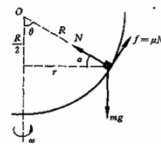


图2-5-2

120 -- EXAMPLE Ch2_Q5

121 -- Q: A hollow spherical shell of radius R rotates around its vertical diameter with angular velocity ω .

122 -- Inside the shell, at height $h = R/2$, a small block rotates together with the shell.

123 -- Let μ be the coefficient of static friction between the block and the shell.

124 -- Prove that the minimum μ required to maintain circular motion at height h

125 -- is $\mu = ((g - R\omega^2 \cos \theta) \sin \theta) / (g \cos \theta + R\omega^2 \sin^2 \theta)$,

126 -- where θ is the angle between the vertical and the line from the shell center to the block, and $r = R \sin \theta$.

127 theorem Ch2_Q5

128 (m : Mass)(R h : Length)(θ : ℝ)

129 (ω : Scalar (-time_unit))

130 (N f : Force)(r : Length)(μ : ℝ)

131 (hθ_range : 0 < θ ∧ θ < Real.pi / 2)

132 (h_pos : 0 < m.val ∧ 0 < R.val ∧ 0 < ω.val ∧ 0 < g)

133 (h_radius : r = Real.cos θ * R)

134 (h_vert : Real.sin θ * N + Real.cos θ * f = m * g)

135 (h_horiz : Real.cos θ * N - Real.sin θ * f = m * ω ** 2 * r)

136 (h_fric_min : f = μ * N):

137 μ = (Real.cos θ * (g - Real.sin θ * R * ω ** 2 / 1)).val /

138 (Real.sin θ * g / 1 + (Real.cos θ) ^ 2 * R * ω ** 2).val := by sorry

(a) Formal Data Construction

Natural Language Statement

Condition

A hollow spherical shell of radius R rotates with angular velocity ω . A small block is located inside the shell at height $h = R/2$, with static friction coefficient μ between the block and the shell. Let θ be the angle between the vertical and the line from the shell's center to the block, and let $r = R \sin \theta$.

Question

What is the minimum coefficient of friction required to keep the block in circular motion?

Steps

We have $r = R \cos \theta$. Consider forces parallel and horizontal to the surfaces, we get:

$$\sin \theta \cdot N + \cos \theta \cdot f = mg \quad (1)$$

$$\cos \theta \cdot N - \sin \theta \cdot f = m\omega^2 r \quad (2)$$

Solve the above equations to get:

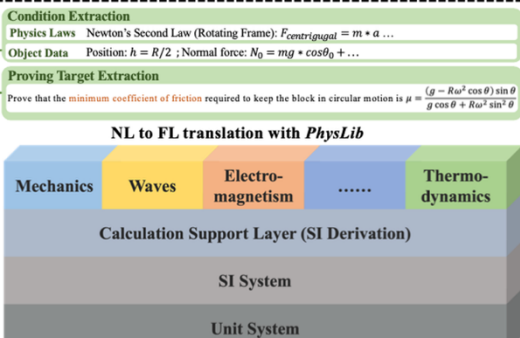
$$N = m(g \sin \theta + \cos^2 \theta \omega^2 R) \quad (3)$$

$$f = m \cos \theta (g - \sin \theta \omega^2 R) \quad (4)$$

Substitute f , N in $\mu = f/N$ and simplify.

$$\mu = \frac{(g - R\omega^2 \cos \theta) \sin \theta}{g \cos \theta + R\omega^2 \sin^2 \theta}$$

Answer



Formal Language Statement

Hypothesis

theorem Ch2_Q5

(m : Mass)(R h : Length)(θ : ℝ)

(ω : Scalar (-time_unit))

(N f : Force)(r : Length)(μ : ℝ)

(hθ_range : 0 < θ ∧ θ < Real.pi / 2)

(h_pos : 0 < m.val ∧ 0 < R.val ∧ 0 < ω.val ∧ 0 < g)

(h_radius : r = Real.cos θ * R)

(h_vert : Real.sin θ * N + Real.cos θ * f = m * g)

(h_horiz : Real.cos θ * N - Real.sin θ * f = m * ω ** 2 * r)

= m * ω ** 2 * r

(h_fric_min : f = μ * N):

μ = (Real.cos θ * (g - Real.sin θ * R * ω ** 2 / 1)).val

(Real.sin θ * g / 1 + (Real.cos θ) ^ 2 * R * ω ** 2).val := by sorry

LeanPhysBench

Optics

Waves

Mechanics

Modern Physics

Thermodynamics

Electromagnetism

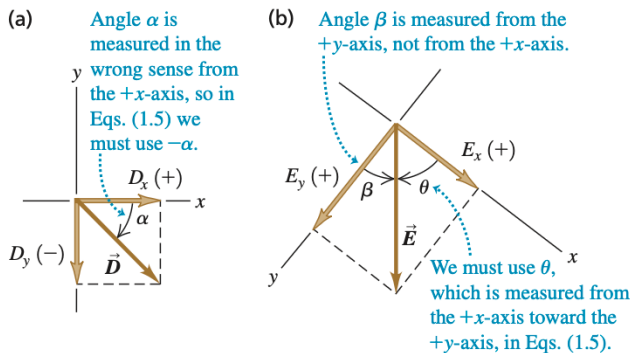
College-Level Examples

EXAMPLE 1.6 Finding components

(a) What are the x - and y -components of vector $\vec{D}$ in Fig. 1.18a? The magnitude of the vector is $D = 3.00$ m, and angle $\alpha = 45^\circ$. (b) What are the x - and y -components of vector $\vec{E}$ in Fig. 1.18b? The magnitude of the vector is $E = 4.50$ m, and angle $\beta = 37.0^\circ$.

IDENTIFY and SET UP We can use Eqs. (1.5) to find the components of these vectors, but we must be careful: Neither angle α nor β in Fig. 1.18 is measured from the $+x$ -axis toward the $+y$ -axis. We estimate from the figure that the lengths of both components in part (a) are roughly 2 m, and that those in part (b) are 3 m and 4 m. The figure indicates the signs of the components.

Figure 1.18 Calculating the x - and y -components of vectors.



EXECUTE (a) The angle α (the Greek letter alpha) between the positive x -axis and $\vec{D}$ is measured toward the *negative* y -axis. The angle we must use in Eqs. (1.5) is $\theta = -\alpha = -45^\circ$. We then find

$$D_x = D \cos \theta = (3.00 \text{ m})(\cos(-45^\circ)) = +2.1 \text{ m}$$

$$D_y = D \sin \theta = (3.00 \text{ m})(\sin(-45^\circ)) = -2.1 \text{ m}$$

Had we carelessly substituted $+45^\circ$ for θ in Eqs. (1.5), our result for D_y would have had the wrong sign.

(b) The x - and y -axes in Fig. 1.18b are at right angles, so it doesn't matter that they aren't horizontal and vertical, respectively. But we can't use the angle β (the Greek letter beta) in Eqs. (1.5), because β is measured from the $+y$ -axis. Instead, we must use the angle $\theta = 90.0^\circ - \beta = 90.0^\circ - 37.0^\circ = 53.0^\circ$. Then we find

$$E_x = E \cos 53.0^\circ = (4.50 \text{ m})(\cos 53.0^\circ) = +2.71 \text{ m}$$

$$E_y = E \sin 53.0^\circ = (4.50 \text{ m})(\sin 53.0^\circ) = +3.59 \text{ m}$$

EVALUATE Our answers to both parts are close to our predictions. But why do the answers in part (a) correctly have only two significant figures?

KEYCONCEPT When you are finding the components of a vector, always use a diagram of the vector and the coordinate axes to guide your calculations.

```

111 -- Question 9
112 -- (a) What are the x- and y-components of vector D
113 -- The magnitude of the vector is D = 3.00 m, and angle a = 45°
114 theorem Mechanics_9_University
115 (x y : Length)(d : Length)(θ: PhysAngle)
116 (hd: d = 3 * meter)
117 (hθ: θ = (π/4) * radian)
118 (hx: x = cos (Scalar.val θ) * d )
119 (hy: y = sin (Scalar.val θ) * d ):
120 (x = (2.1 : Q) * meter ∧ y = (2.1 : Q) * meter) := by sorry
121
122 -- Question 10
123 -- (b) What are the x- and y-components of vector ES
124 -- The magnitude of the vector is E = 4.50 m, and angle b = 37.0°
125 theorem Mechanics_10_University
126 (x y : Length)(e : Length)(θ: PhysAngle)
127 (he: e = (4.5 : Q) * meter)
128 (hθ: θ = (37 * π /180) * radian)
129 (hx: x = cos (Scalar.val θ) * e )
130 (hy: y = sin (Scalar.val θ) * e ):
131 (x = (3.59 : Q) * meter ∧ y = (2.71 : Q) * meter) := by sorry

```

Figure 2.11 A Grand Prix car at two points on the straightaway.



In Eq. (2.5) a_x is the x -component of the acceleration vector, which we call the **instantaneous x -acceleration**; in straight-line motion, all other components of this vector are zero. From now on, when we use the term “acceleration,” we’ll always mean instantaneous acceleration, not average acceleration.

EXAMPLE 2.3 Average and instantaneous accelerations

Suppose the x -velocity v_x of the car in Fig. 2.11 at any time t is given by the equation

$$v_x = 60 \text{ m/s} + (0.50 \text{ m/s}^3)t^2$$

(a) Find the change in x -velocity of the car in the time interval $t_1 = 1.0 \text{ s}$ to $t_2 = 3.0 \text{ s}$. (b) Find the average x -acceleration in this time interval. (c) Find the instantaneous x -acceleration at time $t_1 = 1.0 \text{ s}$ by taking Δt to be first 0.1 s , then 0.01 s , then 0.001 s . (d) Derive an expression for the instantaneous x -acceleration as a function of time, and use it to find a_x at $t = 1.0 \text{ s}$ and $t = 3.0 \text{ s}$.

IDENTIFY and SET UP This example is analogous to Example 2.1 in Section 2.2. In that example we found the average x -velocity from the change in position over shorter and shorter time intervals, and we obtained an expression for the instantaneous x -velocity by differentiating the position as a function of time. In this example we have an exact parallel. Using Eq. (2.4), we’ll find the average x -acceleration from the change in x -velocity over a time interval. Likewise, using Eq. (2.5), we’ll obtain an expression for the instantaneous x -acceleration by differentiating the x -velocity as a function of time.

EXECUTE (a) Before we can apply Eq. (2.4), we must find the x -velocity at each time from the given equation. At $t_1 = 1.0 \text{ s}$ and $t_2 = 3.0 \text{ s}$, the velocities are

$$\begin{aligned} v_{1x} &= 60 \text{ m/s} + (0.50 \text{ m/s}^3)(1.0 \text{ s})^2 = 60.5 \text{ m/s} \\ v_{2x} &= 60 \text{ m/s} + (0.50 \text{ m/s}^3)(3.0 \text{ s})^2 = 64.5 \text{ m/s} \end{aligned}$$

The change in x -velocity Δv_x between $t_1 = 1.0 \text{ s}$ and $t_2 = 3.0 \text{ s}$ is

$$\Delta v_x = v_{2x} - v_{1x} = 64.5 \text{ m/s} - 60.5 \text{ m/s} = 4.0 \text{ m/s}$$

(b) The average x -acceleration during this time interval of duration $t_2 - t_1 = 2.0 \text{ s}$ is

$$a_{\text{av-}x} = \frac{v_{2x} - v_{1x}}{t_2 - t_1} = \frac{4.0 \text{ m/s}}{2.0 \text{ s}} = 2.0 \text{ m/s}^2$$

During this time interval the x -velocity and average x -acceleration have the same algebraic sign (in this case, positive), and the car speeds up.

(c) When $\Delta t = 0.1 \text{ s}$, we have $t_2 = 1.1 \text{ s}$. Proceeding as before, we find

$$\begin{aligned} v_{2x} &= 60 \text{ m/s} + (0.50 \text{ m/s}^3)(1.1 \text{ s})^2 = 60.605 \text{ m/s} \\ \Delta v_x &= 0.105 \text{ m/s} \\ a_{\text{av-}x} &= \frac{\Delta v_x}{\Delta t} = \frac{0.105 \text{ m/s}}{0.1 \text{ s}} = 1.05 \text{ m/s}^2 \end{aligned}$$

You should follow this pattern to calculate $a_{\text{av-}x}$ for $\Delta t = 0.01 \text{ s}$ and $\Delta t = 0.001 \text{ s}$; the results are $a_{\text{av-}x} = 1.005 \text{ m/s}^2$ and $a_{\text{av-}x} = 1.0005 \text{ m/s}^2$, respectively. As Δt gets smaller, the average x -acceleration gets closer to 1.0 m/s^2 , so the instantaneous x -acceleration at $t = 1.0 \text{ s}$ is 1.0 m/s^2 .

(d) By Eq. (2.5) the instantaneous x -acceleration is $a_x = dv_x/dt$. The derivative of a constant is zero and the derivative of t^2 is $2t$, so

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}[60 \text{ m/s} + (0.50 \text{ m/s}^3)t^2] = (0.50 \text{ m/s}^3)(2t) = (1.0 \text{ m/s}^3)t$$

When $t = 1.0 \text{ s}$,

$$a_x = (1.0 \text{ m/s}^3)(1.0 \text{ s}) = 1.0 \text{ m/s}^2$$

When $t = 3.0 \text{ s}$,

$$a_x = (1.0 \text{ m/s}^3)(3.0 \text{ s}) = 3.0 \text{ m/s}^2$$

EVALUATE Neither of the values we found in part (d) is equal to the average x -acceleration found in part (b). That’s because the car’s instantaneous x -acceleration varies with time. The rate of change of acceleration with time is sometimes called the “jerk.”

KEYCONCEPT To calculate an object’s instantaneous acceleration (its average acceleration over an infinitesimally short time interval), take the derivative of its velocity with respect to time.

```

376 -- Question 21
377 -- Suppose the x-velocity v_x, of the car at any time t is given by the equation
378 -- v_x = 60 m/s + (0.50 m/s^3) t^2
379 -- (a) Find the change in x-velocity of the car in the time interval t_1 = 1.0 s t_2 = 3.0 s.
380 -- Answer:
381 -- v_1 = 60 m/s + (0.50 m/s^3)(1.0 s)^2 = 60.5 m/s
382 -- v_2 = 60 m/s + (0.50 m/s^3)(3.0 s)^2 = 64.5 m/s
383 -- delta_v_x = v_2 - v_1 = 64.5 m/s - 60.5 m/s = 4.0 m/s
384 theorem Mechanics_21_University
385 (t_1 t_2 dt : Time)(v_1 v_2 dv: Speed)(xf: Time -> Speed)
386 (ht1 : t_1 = 1 * second)
387 (ht2 : t_2 = 3 * second)
388 (ht : dt = t_2 - t_1)
389 (hvx: ∀ t, xf t = 60 * meter / second + ((0.50:Q) * meter / second**3)* t**2)
390 (hv1: v_1 = xf t_1)
391 (hv2: v_2 = xf t_2)
392 (hdv: dv = v_2 - v_1):
393 (dv = (4 : Q) * meter / second) := by sorry

```

EXAMPLE 2.6 A freely falling coin

A one-euro coin is dropped from the Leaning Tower of Pisa and falls freely from rest. What are its position and velocity after 1.0 s, 2.0 s, and 3.0 s? Ignore air resistance.

IDENTIFY and SET UP “Falls freely” means “falls with constant acceleration due to gravity,” so we can use the constant-acceleration equations. The right side of Fig. 2.23 shows our motion diagram for the coin. The motion is vertical, so we use a vertical coordinate axis and call the coordinate y instead of x . We take the origin at the starting point and the *upward* direction as positive. Both the initial coordinate y_0 and initial y -velocity v_{0y} are zero. The y -acceleration is downward (in the negative y -direction), so $a_y = -g = -9.8 \text{ m/s}^2$. (Remember

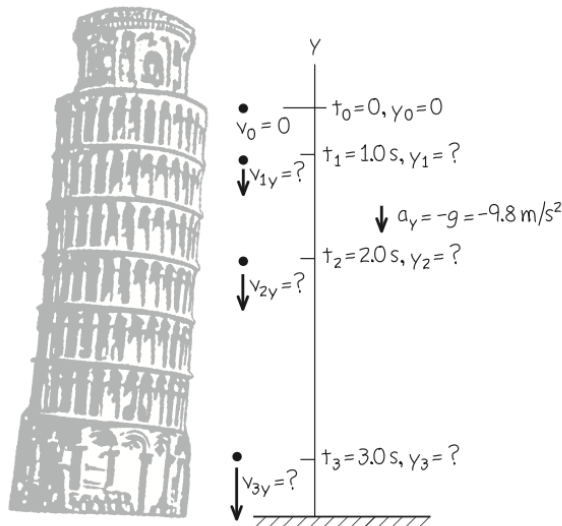
that g is a positive quantity.) Our target variables are the values of y and v_y at the three given times. To find these, we use Eqs. (2.12) and (2.8) with x replaced by y . Our choice of the upward direction as positive means that all positions and velocities we calculate will be negative.

EXECUTE At a time t after the coin is dropped, its position and y -velocity are

$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 = 0 + 0 + \frac{1}{2}(-g)t^2 = (-4.9 \text{ m/s}^2)t^2$$

$$v_y = v_{0y} + a_y t = 0 + (-g)t = (-9.8 \text{ m/s}^2)t$$

Figure 2.23 A coin freely falling from rest.



When $t = 1.0 \text{ s}$, $y = (-4.9 \text{ m/s}^2)(1.0 \text{ s})^2 = -4.9 \text{ m}$ and $v_y = (-9.8 \text{ m/s}^2)(1.0 \text{ s}) = -9.8 \text{ m/s}$; after 1.0 s, the coin is 4.9 m below the origin (y is negative) and has a downward velocity (v_y is negative) with magnitude 9.8 m/s.

We can find the positions and y -velocities at 2.0 s and 3.0 s in the same way. The results are $y = -20 \text{ m}$ and $v_y = -20 \text{ m/s}$ at $t = 2.0 \text{ s}$, and $y = -44 \text{ m}$ and $v_y = -29 \text{ m/s}$ at $t = 3.0 \text{ s}$.

EVALUATE All our answers are negative, as we expected. If we had chosen the positive y -axis to point downward, the acceleration would have been $a_y = +g$ and all our answers would have been positive.

KEYCONCEPT By using one or more of the four equations in Table 2.5 with x replaced by y , the positive y -direction chosen to be upward, and acceleration $a_y = -g$, you can solve any free-fall problem.

```

518 -- Quesiton 28
519 -- A one-euro coin is dropped from the Leaning Tower of Pisa and falls freely from rest.
520 -- What are its position and velocity after 1.0 s, 2.0 s, and 3.0 s? Ignore air resistance.
521 theorem Mechanics_28_University
522 (t_0 t_1 t_2 t_3 dt_1 dt_2 dt_3 t : Time)(x_0 x_1 x_2 x_3 : Length)(v_0 v_1 v_2 v_3 : Speed)(a : Acceleration)
523 (xf xf1: Time -> Length)(vf : Time -> Speed)
524 (ht_0 : t_0 = 0 * second)
525 (ht_1 : t_1 = 1 * second)
526 (ht_2 : t_2 = 2 * second)
527 (ht_3 : t_3 = 3 * second)
528 (ht1 : dt_1 = t_1 - t_0)
529 (ht2 : dt_2 = t_2 - t_0)
530 (ht3 : dt_3 = t_3 - t_0)
531 (hv_0 : v_0 = 0 * meter / second)
532 (ha : a = - g)
533 (hx_0 : x_0 = 0 * meter)
534 (hx : ∀ t, xf t = x_0 + v_0 * t / 1 + (a * t**2)/2)
535 (hv : ∀ t, vf t = v_0 + a * t/1)
536 (hv_1: v_1 = vf t_1)
537 (hx_1: x_1 = xf t_1)
538 (hv_2: v_2 = vf t_2)
539 (hx_2: x_2 = xf t_2)
540 (hv_3: v_3 = vf t_3)
541 (hx_3: x_3 = xf t_3):
542 (x_1 = - (4.9:Q) * meter ∧ v_1 = - (9.8:Q) * meter / second ∧
543 x_2 = - (19.6:Q) * meter ∧ v_2 = - (19.6:Q) * meter / second ∧
544 x_3 = - (44.1:Q) * meter ∧ v_3 = - (29.4:Q) * meter / second) := by sorry

```

You throw a ball vertically upward from the roof of a tall building. The ball leaves your hand at a point even with the roof railing with an upward speed of 15.0 m/s; the ball is then in free fall. (We ignore air resistance.) On its way back down, it just misses the railing. Find (a) the ball's position and velocity 1.00 s and 4.00 s after leaving your hand; (b) the ball's velocity when it is 5.00 m above the railing; (c) the maximum height reached; (d) the ball's acceleration when it is at its maximum height.

IDENTIFY and SET UP The words “in free fall” mean that the acceleration is due to gravity, which is constant. Our target variables are position [in parts (a) and (c)], velocity [in parts (a) and (b)], and acceleration [in part (d)]. We take the origin at the point where the ball leaves your hand, and take the positive direction to be upward (Fig. 2.24). The initial position y_0 is zero, the initial y -velocity v_{0y} is +15.0 m/s, and the y -acceleration is $a_y = -g = -9.80 \text{ m/s}^2$. In part (a), as in Example 2.6, we'll use Eqs. (2.12) and (2.8) to find the position and velocity as functions of time. In part (b) we must find the velocity at a given position (no time is given), so we'll use Eq. (2.13).

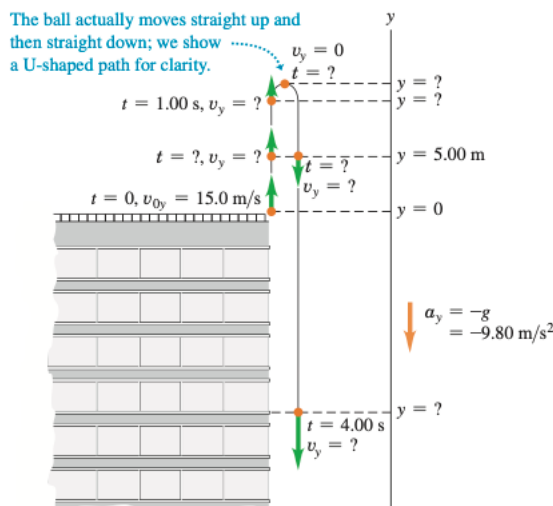
Figure 2.25 (next page) shows the y - t and v_y - t graphs for the ball. The y - t graph is a concave-down parabola that rises and then falls, and the v_y - t graph is a downward-sloping straight line. Note that the ball's velocity is zero when it is at its highest point.

EXECUTE (a) The position and y -velocity at time t are given by Eqs. (2.12) and (2.8) with x 's replaced by y 's:

$$\begin{aligned} y &= y_0 + v_{0y}t + \frac{1}{2}a_y t^2 = y_0 + v_{0y}t + \frac{1}{2}(-g)t^2 \\ &= (0) + (15.0 \text{ m/s})t + \frac{1}{2}(-9.80 \text{ m/s}^2)t^2 \\ v_y &= v_{0y} + a_y t = v_{0y} + (-g)t \\ &= 15.0 \text{ m/s} + (-9.80 \text{ m/s}^2)t \end{aligned}$$

When $t = 1.00 \text{ s}$, these equations give $y = +10.1 \text{ m}$ and $v_y = +5.2 \text{ m/s}$. That is, the ball is 10.1 m above the origin (y is positive) and

Figure 2.24 Position and velocity of a ball thrown vertically upward.

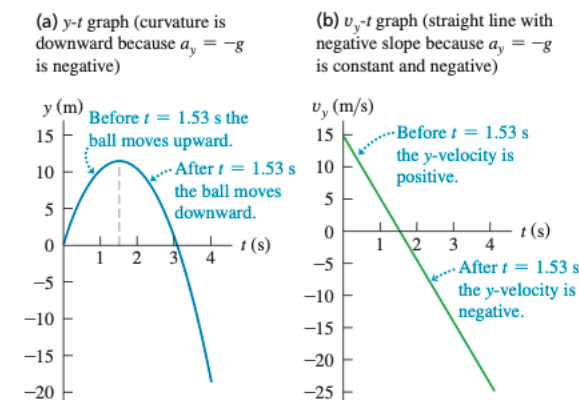


moving upward (v_y is positive) with a speed of 5.2 m/s. This is less than the initial speed because the ball slows as it ascends. When $t = 4.00 \text{ s}$, those equations give $y = -18.4 \text{ m}$ and $v_y = -24.2 \text{ m/s}$. The ball has passed its highest point and is 18.4 m below the origin (y is negative). It is moving downward (v_y is negative) with a speed of 24.2 m/s.

Continued

80 CHAPTER 2 Motion Along a Straight Line

Figure 2.25 (a) Position and (b) velocity as functions of time for a ball thrown upward with an initial speed of 15.0 m/s.



(b) The y -velocity at any position y is given by Eq. (2.13) with x 's replaced by y 's:

$$\begin{aligned} v_y^2 &= v_{0y}^2 + 2a_y(y - y_0) = v_{0y}^2 + 2(-g)(y - 0) \\ &= (15.0 \text{ m/s})^2 + 2(-9.80 \text{ m/s}^2)y \end{aligned}$$

When the ball is 5.00 m above the origin we have $y = +5.00 \text{ m}$, so

$$\begin{aligned} v_y^2 &= (15.0 \text{ m/s})^2 + 2(-9.80 \text{ m/s}^2)(5.00 \text{ m}) = 127 \text{ m}^2/\text{s}^2 \\ v_y &= \pm 11.3 \text{ m/s} \end{aligned}$$

We get *two* values of v_y because the ball passes through the point $y = +5.00 \text{ m}$ twice, once on the way up (so v_y is positive) and once on the way down (so v_y is negative) (see Fig. 2.24). Note that the speed of the ball is 11.3 m/s each time it passes through this point.

(c) At the instant at which the ball reaches its maximum height y_1 , its y -velocity is momentarily zero: $v_y = 0$. We use Eq. (2.13) to find y_1 . With $v_y = 0$, $y_0 = 0$, and $a_y = -g$, we get

$$\begin{aligned} 0 &= v_{0y}^2 + 2(-g)(y_1 - 0) \\ y_1 &= \frac{v_{0y}^2}{2g} = \frac{(15.0 \text{ m/s})^2}{2(9.80 \text{ m/s}^2)} = +11.5 \text{ m} \end{aligned}$$

(d) **CAUTION A free-fall misconception** It's a common misconception that at the highest point of free-fall motion, where the velocity is zero, the acceleration is also zero. If this were so, once the ball reached the highest point it would hang there suspended in midair! Remember that acceleration is the rate of change of velocity, and the ball's velocity is continuously changing. At every point, including the highest point, and at any velocity, including zero, the acceleration in free fall is always $a_y = -g = -9.80 \text{ m/s}^2$.

EVALUATE A useful way to check any free-fall problem is to draw the y - t and v_y - t graphs, as we did in Fig. 2.25. Note that these are graphs of Eqs. (2.12) and (2.8), respectively. Given the initial position, initial velocity, and acceleration, you can easily create these graphs by using a graphing calculator app or an online math program.

KEYCONCEPT If a freely falling object passes a given point at two different times, once moving upward and once moving downward, its speed will be the same at both times.

```

546 -- Quesiton 29
547 -- You throw a ball vertically upward from the roof of a tall building.
548 -- The ball leaves your hand at a point even with the roof railing withan upward speed of 15.0 m/s;
549 -- the ball is then in free fall. (We ignoreair resistance,) On its way back down, it just misses the railing.
550 -- (a) the ball's position and velocity 1.00 s and 4.00 s after leaving yourhand;
551 theorem Mechanics_29_University
552   (t_0 t_1 t_2 t : Time)(x_0 x_1 x_2 : Length)(v_0 v_1 v_2 : Speed)(a : Acceleration)
553   (xf xf1: Time → Length)(vf : Time → Speed)
554   (ht_0 : t_0 = 0 • second)
555   (ht_1 : t_1 = 1 • second)
556   (ht_2 : t_2 = 4 • second)
557   (ht1 : t = t_1 - t_0)
558   (ht2 : dt = t_2 - t_0)
559   (hv_0 : v_0 = 15 • meter / second)
560   (ha : a = - g)
561   (hx_0 : x_0 = 0 • meter)
562   (hx : ∀ t, xf t = x_0 + v_0 * t / 1 + (a * t**2)/2)
563   (hv : ∀ t, vf t = v_0 + a * t/1)
564   (hv_1: v_1 = vf t_1)
565   (hx_1: x_1 = xf t_1)
566   (hv_2: v_2 = vf t_2)
567   (hx_2: x_2 = xf t_2):
568   (x_1 = (10.1:Q) • meter ∧ v_1 = (5.2:Q) • meter / second ∧
569     x_2 = - (18.4:Q) • meter ∧ v_2 = - (24.2:Q) • meter / second) := by sorry

```



```

571 -- Quesiton 30
572 -- You throw a ball vertically upward from the roof of a tall building.
573 -- The ball leaves your hand at a point even with the roof railing withan upward speed of 15.0 m/s;
574 -- the ball is then in free fall. (We ignoreair resistance,) On its way back down, it just misses the railing.
575 -- (b) the ball's velocity when it is 5.00 m above the railing;
576 theorem Mechanics_30_University
577   (t_0 t_1 t : Time)(x_0 x_1 : Length)(v_0 v_1 : Speed)(a : Acceleration)
578   (xf xf1: Time → Length)(vf : Time → Speed)
579   (ht_0 : t_0 = 0 • second)
580   (hv_0 : v_0 = 15 • meter / second)
581   (ha : a = - g)
582   (hx_0 : x_0 = 0 • meter)
583   (hx : ∀ t, xf t = x_0 + v_0 * t / 1 + (a * t**2)/2)
584   (hv : ∀ t, vf t = v_0 + a * t/1)
585   (hx_1: x_1 = xf t)
586   (hx_x: x_1 = 5 • meter)
587   (hv_1: v_1 = vf t):
588   (v_1 = (11.3:Q) • meter / second ∧ v_1 = - (11.3:Q) • meter / second) := by sorry

```

EXAMPLE 15.1 Wavelength of a musical sound

WITH  **ARIATION PROBLEMS**

Sound waves are longitudinal waves in air. The speed of sound depends on temperature; at 20°C it is 344 m/s. What is the wavelength of a sound wave in air at 20°C if the frequency is 262 Hz (the approximate frequency of middle C on a piano)?

IDENTIFY and SET UP This problem involves Eq. (15.1), $v = \lambda f$, which relates wave speed v , wavelength λ , and frequency f for a periodic wave. The target variable is the wavelength λ . We are given $v = 344$ m/s and $f = 262$ Hz = 262 s⁻¹.

EXECUTE We solve Eq. (15.1) for λ :

$$\lambda = \frac{v}{f} = \frac{344 \text{ m/s}}{262 \text{ Hz}} = \frac{344 \text{ m/s}}{262 \text{ s}^{-1}} = 1.31 \text{ m}$$

EVALUATE The speed v of sound waves does *not* depend on the frequency. Hence $\lambda = v/f$ says that wavelength changes in inverse proportion to frequency. As an example, high (soprano) C is two octaves above middle C. Each octave corresponds to a factor of 2 in frequency, so the frequency of high C is four times that of middle C: $f = 4(262 \text{ Hz}) = 1048 \text{ Hz}$. Hence the *wavelength* of high C is *one-fourth* as large: $\lambda = (1.31 \text{ m})/4 = 0.328 \text{ m}$.

KEYCONCEPT The product of a wave's wavelength and frequency has the same value no matter what the frequency is. This product equals the wave speed.

```

16 -- Question 1
17 -- Sound waves are longitudinal waves in air.
18 -- The speed of sound depends on temperature; at 20°C it is 344 m/s.
19 -- What is the wavelength of a sound wave in air at 20°C if the frequency is 262 Hz
20 -- (the approximate frequency of middle C on a piano)?
21 theorem Waves_1_University
22   (h :Length)(v : Speed)(f : Frequency)
23   (h_v: v = 344 • meter / second)
24   (h_f: f = 262 • hertz )
25   (hh: h = v / f):
26   h = (344/262 : Q) • meter := by sorry

```

EXAMPLE 15.5 The inverse-square law

WITH  **ARIATION PROBLEMS**

A siren on a tall pole radiates sound waves uniformly in all directions. At a distance of 15.0 m from the siren, the sound intensity is 0.250 W/m². At what distance is the intensity 0.010 W/m²?

IDENTIFY and SET UP Because sound is radiated uniformly in all directions, we can use the inverse-square law, Eq. (15.26). At $r_1 = 15.0$ m the intensity is $I_1 = 0.250$ W/m², and the target variable is the distance r_2 at which the intensity is $I_2 = 0.010$ W/m².

EXECUTE We solve Eq. (15.26) for r_2 :

$$r_2 = r_1 \sqrt{\frac{I_1}{I_2}} = (15.0 \text{ m}) \sqrt{\frac{0.250 \text{ W/m}^2}{0.010 \text{ W/m}^2}} = 75.0 \text{ m}$$

EVALUATE As a check on our answer, note that r_2 is five times greater than r_1 . By the inverse-square law, the intensity I_2 should be $1/5^2 = 1/25$ as great as I_1 , and indeed it is.

By using the inverse-square law, we've assumed that the sound waves travel in straight lines away from the siren. A more realistic solution, which is beyond our scope, would account for the reflection of sound waves from the ground.

KEYCONCEPT If waves emitted from a source spread out equally in all directions, the wave intensity is inversely proportional to the square of the distance from the source.

```

157 -- Question 10
158 -- A siren on a tall pole radiates sound waves uniformly in all directions.
159 -- At a distance of 15.0 m from the siren, the sound intensity is 0.250 W/m^2.
160 -- At What distance is th intensity 0.010 W/m^2?
161 theorem Waves_10_University
162   (d1 d2:Length)(P1 P2:Scalar (power_unit~2• length_unit))
163   (hd1:d1=15•meter)
164   (hp1:P1=(0.25:R) • StandardUnit _)
165   (hp2:P2=(0.01:R) • StandardUnit _)
166   (hd1:P1*d1**2=P2*d2**2):
167   d2=75• meter
168   :=by
169   sorry

```

EXAMPLE 17.1 Body temperature

You jump into a pool at $T_1 = 25.00^\circ\text{C}$ and your body temperature is $T_2 = 37.00^\circ\text{C}$. Express these temperatures in kelvins, and find $\Delta T = T_2 - T_1$ both in degrees Celsius and kelvins.

IDENTIFY and SET UP Our target variables are stated above. We convert Celsius temperatures to Kelvin by using Eq. (17.1).

EXECUTE Since $T_1 = 25.00^\circ\text{C}$ and $T_2 = 37.00^\circ\text{C}$, $\Delta T = T_2 - T_1 = 12.00^\circ\text{C}$. To get the Kelvin temperatures, just add 273.15 to each Celsius temperature: $T_1 = 298.15\text{ K}$ and $T_2 = 310.15\text{ K}$. The temperature difference is $\Delta T = T_2 - T_1 = 12.00\text{ K}$.

EVALUATE The Celsius and Kelvin scales have different zero points but the same size degrees. Therefore *any* temperature difference ΔT is the *same* on the Celsius and Kelvin scales.

KEYCONCEPT The Celsius and Kelvin scales have different zero points, but differences in temperature are the same in both scales.

```
17  -- Question 1 celsius kelvin unsolved
18  -- You jump into a pool at T_1= 25.00°C and your body temperature is T_2= 37.00°C.
19  -- Express these temperatures in kelvins, and find delta_T = T_2 - T_1 both in degrees Celsius and kelvins.
20  theorem Thermodynamics_1_University
21  (T_1K T_2K T_1C T_2C dT_K dT_C dKC : Temperature)
22  (hd: dKC = (27315/100 : Q) * kelvin)
23  (hT1C: T_1C = 25 * kelvin)
24  (hT2C: T_2C = 37 * kelvin)
25  (hT1K: T_1K = T_1C + dKC)
26  (hT2K: T_2K = T_2C + dKC)
27  (hdT: dT_K = T_2K - T_1K)
28  (hdTC: dT_C = T_2C - T_1C):
29  dT_K = dT_C := by
30  simp [←Scalar.val_inj, hd, hT1C, hT2C, hT1K, hT2K, hdT, hdTC, kelvin]
```

EXAMPLE 17.3 Volume change due to temperature change

WITH  **VARIATION PROBLEMS**

A 200 cm^3 glass flask is filled to the brim with mercury at 20°C . How much mercury overflows when the temperature of the system is raised to 100°C ? The coefficient of *linear* expansion of the glass is $0.40 \times 10^{-5}\text{ K}^{-1}$.

IDENTIFY and SET UP This problem involves the volume expansion of the glass and of the mercury. The amount of overflow depends on the *difference* between the volume changes ΔV for these two materials, both given by Eq. (17.6). The mercury will overflow if its coefficient of volume expansion β (see Table 17.2) is greater than that of glass, which we find from Eq. (17.7) using the given value of α .

EXECUTE From Table 17.2, $\beta_{\text{Hg}} = 18 \times 10^{-5}\text{ K}^{-1}$. That is indeed greater than $\beta_{\text{glass}} = 3\alpha_{\text{glass}} = 3(0.40 \times 10^{-5}\text{ K}^{-1}) = 1.2 \times 10^{-5}\text{ K}^{-1}$, from Eq. (17.7). The volume overflow is then

$$\begin{aligned}\Delta V_{\text{Hg}} - \Delta V_{\text{glass}} &= \beta_{\text{Hg}} V_0 \Delta T - \beta_{\text{glass}} V_0 \Delta T \\ &= V_0 \Delta T (\beta_{\text{Hg}} - \beta_{\text{glass}}) \\ &= (200\text{ cm}^3)(80^\circ\text{C})(18 \times 10^{-5} - 1.2 \times 10^{-5}) = 2.7\text{ cm}^3\end{aligned}$$

EVALUATE This is basically how a mercury-in-glass thermometer works; the column of mercury inside a sealed tube rises as T increases because mercury expands faster than glass.

As Tables 17.1 and 17.2 show, glass has smaller coefficients of expansion α and β than do most metals. This is why you can use hot water to loosen a metal lid on a glass jar; the metal expands more than the glass does.

KEYCONCEPT A change in temperature causes the *volume* of an object to change by an amount that is approximately proportional to the object's initial volume and to the temperature change ΔT .

```
60  -- Question 4
61  -- A 200 cm^3 glass flask is filled to the brim with mercury at 20°C.
62  -- How much mercury overflows when the temperature of the system is raised to 100°C?
63  -- The coefficient of linear expansion of the glass is 0.40 * 10^(-5) * K^(-1)
64  theorem Thermodynamics_4_University
65  (T_1C T_2C dT : Temperature)(V_20C V_100C dV: Volume)
66  (hT1C: T_1C = 20 * kelvin)
67  (hT2C: T_2C = 100 * kelvin)
68  (hdT: dT = T_2C - T_1C)
69  (hV20C: V_20C = (200/1000000 : Q) * meter**3)
70  (hdV: dV = ((18-3*0.4):Q) * V_20C * dT / (100000 * kelvin))
71  (hV100C: V_100C = V_20C - dV):
72  dV = (27/1000000 : Q) * meter**3 := by sorry
```

EXAMPLE 17.10 Combustion, temperature change, and phase change

In a particular camp stove, only 30% of the energy released in burning petrol goes to heating the water in a pot on the stove. How much petrol must we burn to heat 1.00 L (1.00 kg) of water from 20°C to 100°C and boil away 0.25 kg of it?

IDENTIFY and SET UP All of the water undergoes a temperature change and part of it undergoes a phase change, from liquid to gas. We determine the heat required to cause both of these changes, and then use the 30% combustion efficiency to determine the amount of petrol that must be burned (the target variable). We use Eqs. (17.11) and (17.18) and the idea of heat of combustion.

EXECUTE To raise the temperature of the water from 20°C to 100°C requires

$$Q_1 = mc \Delta T = (1.00 \text{ kg})(4190 \text{ J/kg} \cdot \text{K})(80 \text{ K}) = 3.35 \times 10^5 \text{ J}$$

To boil 0.25 kg of water at 100°C requires

$$Q_2 = mL_v = (0.25 \text{ kg})(2.256 \times 10^6 \text{ J/kg}) = 5.64 \times 10^5 \text{ J}$$

The total energy needed is $Q_1 + Q_2 = 8.99 \times 10^5 \text{ J}$. This is 30% = 0.30 of the total heat of combustion, which is therefore $(8.99 \times 10^5 \text{ J})/0.30 = 3.00 \times 10^6 \text{ J}$. As we mentioned earlier, the combustion of 1 g of petrol releases 46,000 J, so the mass of petrol required is $(3.00 \times 10^6 \text{ J})/(46,000 \text{ J/g}) = 65 \text{ g}$, or a volume of about 0.09 L of petrol.

EVALUATE This result suggests the tremendous amount of energy released in burning even a small quantity of petrol. Another 123 g of petrol would be required to boil away the remaining water; can you prove this?

KEYCONCEPT To find the amount of heat released when a quantity of substance undergoes combustion, multiply the mass that combusts by the heat of combustion of the substance.

```
165 -- Question 11
166 -- In a particular camp stove, only 30% of the energy released in burning petrol goes to heating the water in a pot on the stove.
167 -- How much petrol must we burn to heat 1.00 L (1.00 kg) of water from 20°C to 100°C and boil away 0.25 kg of it?
168 theorem Thermodynamics_11_University
169 (m1 m2:Mass)(T1 T2 T_eq dT1 dT2:Temperature)(c1 c2:HeatCapacity)(L_f:HeatOfFusion)(Q1 Q2 Q3:Energy)(efficiency:Dimensionless)(E_fuel:Energy)
170 (hm1: m1 = (1000/1000 : Q) * kilogram)
171 (hm2: m2 = (250/1000 : Q) * kilogram)
172 (hT1: T1 = (20 + 27315/100 : Q) * kelvin)
173 (hT2: T2 = (100 + 27315/100 : Q) * kelvin)
174 (hdT: dT = T2 - T1)
175 (hc1: c1 = (4190 : Q) * joule / kilogram / kelvin)
176 (hLf: L_f = (2256000 : Q) * joule / kilogram)
177 (hQ1: Q1 = m1 * c1 * dT)
178 (hQ2: Q2 = m2 * L_f)
179 (hQ: Q1 + Q2 = (3/10 : Q) * E_fuel):
180 E_fuel = (3000000 : Q) * joule := by sorry
```